Borel-de Siebenthal theory for real affine root systems

R. Venkatesh

Department of Mathematics, Indian Institute of Science, Bangalore, India

June 04, 2018

Definitions

- Our base field is complex numbers throughout.
- A subset Ψ of Φ is called a subroot system of Φ if $\mathbf{s}_\alpha(\beta) \in \Psi$ for all $\alpha, \beta \in \Psi$.
- A subroot system Ψ of Φ is called a closed subroot system of Φ if $\alpha, \beta \in \Psi$ and $\alpha + \beta \in \Phi,$ then $\alpha + \beta \in \Psi$.
- A proper closed subroot system Ψ of Φ is said to be a maximal closed subroot system of Φ if for every closed subroot system Δ of Φ the condition $\Psi \subseteq \Delta \subseteq \Phi$ implies that either $\Delta = \Psi$ or $\Delta = \Phi$.

Definitions

- Our base field is complex numbers throughout.
- A subset Ψ of Φ is called a subroot system of Φ if $\mathbf{s}_\alpha(\beta) \in \Psi$ for all $\alpha, \beta \in \Psi$.
- A subroot system Ψ of Φ is called a closed subroot system of Φ if $\alpha, \beta \in \Psi$ and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \Psi$.
- A proper closed subroot system Ψ of Φ is said to be a maximal closed subroot system of Φ if for every closed subroot system Δ of Φ the condition $\Psi \subseteq \Delta \subseteq \Phi$ implies that either $\Delta = \Psi$ or $\Delta = \Phi$.

Definitions

- Our base field is complex numbers throughout.
- A subset Ψ of Φ is called a subroot system of Φ if $\mathbf{s}_\alpha(\beta) \in \Psi$ for all $\alpha, \beta \in \Psi$.
- A subroot system Ψ of Φ is called a closed subroot system of Φ if $\alpha, \beta \in \Psi$ and $\alpha + \beta \in \Phi,$ then $\alpha + \beta \in \Psi$.
- A proper closed subroot system Ψ of Φ is said to be a maximal closed subroot system of Φ if for every closed subroot system Δ of Φ the condition $\Psi \subseteq \Delta \subseteq \Phi$ implies that either $\Delta = \Psi$ or $\Delta = \Phi$.

Definitions

- Our base field is complex numbers throughout.
- A subset Ψ of Φ is called a subroot system of Φ if $\mathbf{s}_\alpha(\beta) \in \Psi$ for all $\alpha, \beta \in \Psi$.
- A subroot system Ψ of Φ is called a closed subroot system of Φ if $\alpha, \beta \in \Psi$ and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \Psi$.
- A proper closed subroot system Ψ of Φ is said to be a maximal closed subroot system of Φ if for every closed subroot system Δ of Φ the condition $\Psi \subseteq \Delta \subseteq \Phi$ implies that either $\Delta = \Psi$ or $\Delta = \Phi$.

Motivation

Classification of maximal closed subroot systems of given root systems is very essential. For example it plays a vital role in the following things:

- Classification of all the maximal closed connected subgroups of maximal rank of a connected compact Lie group. [A. Borel and J. De Siebenthal, Comment. Math. Helv., 1949]
- Classification of all semi-simple subalgebras of finite dimensional complex semi-simple Lie algebras. [E. B. Dynkin [Doklady Akad. Nauk SSSR (N.S.), 1950]
- Classification of all subalgebras of Kac-Moody algebras which is of Kac-Moody type.

Motivation

Classification of maximal closed subroot systems of given root systems is very essential. For example it plays a vital role in the following things:

- Classification of all the maximal closed connected subgroups of maximal rank of a connected compact Lie group. [A. Borel and J. De Siebenthal, Comment. Math. Helv., 1949]
- Classification of all semi-simple subalgebras of finite dimensional complex semi-simple Lie algebras. [E. B. Dynkin [Doklady Akad. Nauk SSSR (N.S.), 1950]
- Classification of all subalgebras of Kac-Moody algebras which is of Kac-Moody type.

Motivation

Classification of maximal closed subroot systems of given root systems is very essential. For example it plays a vital role in the following things:

- Classification of all the maximal closed connected subgroups of maximal rank of a connected compact Lie group. [A. Borel and J. De Siebenthal, Comment. Math. Helv., 1949]
- Classification of all semi-simple subalgebras of finite dimensional complex semi-simple Lie algebras. [E. B. Dynkin [Doklady Akad. Nauk SSSR (N.S.), 1950]
- Classification of all subalgebras of Kac-Moody algebras which is of Kac-Moody type.

Regular subalgebra

- E. B. Dynkin introduced regular subalgebras in order to classify all the semi-simple subalgebras of given finite dimensional semi-simple Lie algebras.
- He classified regular semi-simple subalgebras in terms of their root systems, which are closed subroot systems of the root system of the ambient Lie algebra.
- One can define regular subalgebras in the context of affine Kac–Moody algebras by generalizing the definition of regular semi-simple subalgebras.
- A subalgebra of the affine Kac–Moody algebra $\mathfrak{g}$ is said to be a regular subalgebra if there exists a closed subroot system Ψ of Φ such that it is generated as a Lie subalgebra by the root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \Psi$.

Regular subalgebra

- E. B. Dynkin introduced regular subalgebras in order to classify all the semi-simple subalgebras of given finite dimensional semi-simple Lie algebras.
- He classified regular semi-simple subalgebras in terms of their root systems, which are closed subroot systems of the root system of the ambient Lie algebra.
- One can define regular subalgebras in the context of affine Kac–Moody algebras by generalizing the definition of regular semi-simple subalgebras.
- A subalgebra of the affine Kac–Moody algebra $\mathfrak{g}$ is said to be a regular subalgebra if there exists a closed subroot system Ψ of Φ such that it is generated as a Lie subalgebra by the root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \Psi$.

Regular subalgebra

- E. B. Dynkin introduced regular subalgebras in order to classify all the semi-simple subalgebras of given finite dimensional semi-simple Lie algebras.
- He classified regular semi-simple subalgebras in terms of their root systems, which are closed subroot systems of the root system of the ambient Lie algebra.
- One can define regular subalgebras in the context of affine Kac–Moody algebras by generalizing the definition of regular semi-simple subalgebras.
- A subalgebra of the affine Kac–Moody algebra $\mathfrak{g}$ is said to be a regular subalgebra if there exists a closed subroot system Ψ of Φ such that it is generated as a Lie subalgebra by the root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \Psi$.

Regular subalgebra

- E. B. Dynkin introduced regular subalgebras in order to classify all the semi-simple subalgebras of given finite dimensional semi-simple Lie algebras.
- He classified regular semi-simple subalgebras in terms of their root systems, which are closed subroot systems of the root system of the ambient Lie algebra.
- One can define regular subalgebras in the context of affine Kac–Moody algebras by generalizing the definition of regular semi-simple subalgebras.
- A subalgebra of the affine Kac–Moody algebra $\mathfrak{g}$ is said to be a regular subalgebra if there exists a closed subroot system Ψ of Φ such that it is generated as a Lie subalgebra by the root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \Psi$.

Motivation

- The regular subalgebra defined by Ψ is uniquely determined by Ψ and conversely Ψ is also uniquely determined the regular subalgebra defined by Ψ.
- Another motivation for this work comes from the work of A. Felikson, A. Retakh and P. Tumarkin [J. Phys. A, 2008] where they described a procedure to classify all the regular subalgebras of affine Kac–Moody algebras.
- They determine all possible maximal closed affine type root subsystems in terms of their Weyl group in order to classify all the regular subalgebras.

Motivation

- The regular subalgebra defined by Ψ is uniquely determined by Ψ and conversely Ψ is also uniquely determined the regular subalgebra defined by Ψ.
- Another motivation for this work comes from the work of A. Felikson, A. Retakh and P. Tumarkin [J. Phys. A, 2008] where they described a procedure to classify all the regular subalgebras of affine Kac–Moody algebras.
- They determine all possible maximal closed affine type root subsystems in terms of their Weyl group in order to classify all the regular subalgebras.

Motivation

- The regular subalgebra defined by Ψ is uniquely determined by Ψ and conversely Ψ is also uniquely determined the regular subalgebra defined by Ψ.
- Another motivation for this work comes from the work of A. Felikson, A. Retakh and P. Tumarkin [J. Phys. A, 2008] where they described a procedure to classify all the regular subalgebras of affine Kac–Moody algebras.
- They determine all possible maximal closed affine type root subsystems in terms of their Weyl group in order to classify all the regular subalgebras.

Motivation

- But it appears that some maximal subroot systems were omitted in their classification list. For example, the root system of type $\mathtt{A}_2^{(1)} \oplus \mathtt{A}_2^{(1)}$ appears as a decomposable maximal closed subroot system of $\mathtt{E}_6^{(2)}$ and the root system of type $\mathtt{D}_5^{(2)}$ appears as an indecomposable maximal closed subroot system of $\mathtt{E}_6^{(2)}$, which were omitted in their list.
- M. J. Dyer and G. I. Lehrer [Transform. Groups, 2011] developed some new ideas to classify all the subroot systems of untwisted affine root systems, or more generally the subroot systems of real root systems of loop algebras of Kac–Moody algebras.

Motivation

- But it appears that some maximal subroot systems were omitted in their classification list. For example, the root system of type $\mathtt{A}_2^{(1)} \oplus \mathtt{A}_2^{(1)}$ appears as a decomposable maximal closed subroot system of $\mathtt{E}_6^{(2)}$ and the root system of type $\mathtt{D}_5^{(2)}$ appears as an indecomposable maximal closed subroot system of $\mathtt{E}_6^{(2)}$, which were omitted in their list.
- M. J. Dyer and G. I. Lehrer [Transform. Groups, 2011] developed some new ideas to classify all the subroot systems of untwisted affine root systems, or more generally the subroot systems of real root systems of loop algebras of Kac–Moody algebras.

Our work

- Let Φ be an affine root system and let $\Psi \leq \Phi$ a subroot system. The gradient root system associated with Ψ is defined to be $\mathrm{Gr}(\Psi) := \left\{(\alpha + r\delta)|_{\mathring{\mathfrak{h}}} : \alpha + r\delta \in \Psi\right\}$, where $\mathring{\mathfrak{h}}$ is the Cartan subalgebra of underlying semi-simple Lie algebra $\mathring{\mathfrak{g}}$.
- Since $\delta|_{\mathring{\mathfrak{h}}} = 0$, we have $(\alpha + r\delta)|_{\mathring{\mathfrak{h}}} = \alpha|_{\mathring{\mathfrak{h}}} = \alpha$ for $\alpha + r\delta \in \Psi$.
- In particular we have

$$\mathrm{Gr}(\Phi) = \begin{cases} \mathring{\Phi} \cup \frac{1}{2}\mathring{\Phi}_\ell & \text{if } \widehat{\mathfrak{g}} \text{ is of type } \mathrm{A}_{2n}^{(2)} \\ \mathring{\Phi} & \text{otherwise.} \end{cases}$$

  The definition of $\mathrm{Gr}(\Psi)$ is dependent on the ambient root system Φ.
- $Z_\alpha(\Psi) = \{r : \alpha + r\delta \in \Psi\}$, for $\alpha \in \mathrm{Gr}(\Psi)$.

Our work

- Let Φ be an affine root system and let $\Psi \leq \Phi$ a subroot system. The gradient root system associated with Ψ is defined to be $\mathrm{Gr}(\Psi) := \left\{(\alpha + r\delta)|_{\mathring{\mathfrak{h}}} : \alpha + r\delta \in \Psi\right\}$, where $\mathring{\mathfrak{h}}$ is the Cartan subalgebra of underlying semi-simple Lie algebra $\mathring{\mathfrak{g}}$.
- Since $\delta|_{\mathring{\mathfrak{h}}} = 0$, we have $(\alpha + r\delta)|_{\mathring{\mathfrak{h}}} = \alpha|_{\mathring{\mathfrak{h}}} = \alpha$ for $\alpha + r\delta \in \Psi$.
- In particular we have

$$\mathrm{Gr}(\Phi) = \begin{cases} \mathring{\Phi} \cup \frac{1}{2}\mathring{\Phi}_\ell & \text{if } \widehat{\mathfrak{g}} \text{ is of type } \mathrm{A}_{2n}^{(2)} \\ \mathring{\Phi} & \text{otherwise.} \end{cases}$$

  The definition of $\mathrm{Gr}(\Psi)$ is dependent on the ambient root system Φ.
- $Z_\alpha(\Psi) = \{r : \alpha + r\delta \in \Psi\}$, for $\alpha \in \mathrm{Gr}(\Psi)$.

Our work

- Let Φ be an affine root system and let $\Psi \leq \Phi$ a subroot system. The gradient root system associated with Ψ is defined to be $\mathrm{Gr}(\Psi) := \left\{(\alpha + r\delta)|_{\mathring{\mathfrak{h}}} : \alpha + r\delta \in \Psi\right\}$, where $\mathring{\mathfrak{h}}$ is the Cartan subalgebra of underlying semi-simple Lie algebra $\mathring{\mathfrak{g}}$.
- Since $\delta|_{\mathring{\mathfrak{h}}} = 0$, we have $(\alpha + r\delta)|_{\mathring{\mathfrak{h}}} = \alpha|_{\mathring{\mathfrak{h}}} = \alpha$ for $\alpha + r\delta \in \Psi$.
- In particular we have

$$\mathrm{Gr}(\Phi) = \begin{cases} \mathring{\Phi} \cup \frac{1}{2}\mathring{\Phi}_\ell & \text{if } \widehat{\mathfrak{g}} \text{ is of type } \mathtt{A}_{2n}^{(2)} \\ \mathring{\Phi} & \text{otherwise.} \end{cases}$$

  The definition of $\mathrm{Gr}(\Psi)$ is dependent on the ambient root system Φ.
- $Z_\alpha(\Psi) = \{r : \alpha + r\delta \in \Psi\}$, for $\alpha \in \mathrm{Gr}(\Psi)$.

Our work

- Let Φ be an affine root system and let $\Psi \leq \Phi$ a subroot system. The gradient root system associated with Ψ is defined to be $\mathrm{Gr}(\Psi) := \left\{(\alpha + r\delta)|_{\mathring{\mathfrak{h}}} : \alpha + r\delta \in \Psi\right\}$, where $\mathring{\mathfrak{h}}$ is the Cartan subalgebra of underlying semi-simple Lie algebra $\mathring{\mathfrak{g}}$.
- Since $\delta|_{\mathring{\mathfrak{h}}} = 0$, we have $(\alpha + r\delta)|_{\mathring{\mathfrak{h}}} = \alpha|_{\mathring{\mathfrak{h}}} = \alpha$ for $\alpha + r\delta \in \Psi$.
- In particular we have

$$\mathrm{Gr}(\Phi) = \begin{cases} \mathring{\Phi} \cup \frac{1}{2}\mathring{\Phi}_\ell & \text{if } \widehat{\mathfrak{g}} \text{ is of type } \mathtt{A}_{2n}^{(2)} \\ \mathring{\Phi} & \text{otherwise.} \end{cases}$$

  The definition of $\mathrm{Gr}(\Psi)$ is dependent on the ambient root system Φ.
- $Z_\alpha(\Psi) = \{r : \alpha + r\delta \in \Psi\}$, for $\alpha \in \mathrm{Gr}(\Psi)$.

Our work

Proposition (Dyer-Lehrer, KV)

(1) Let Φ be an irreducible affine root system and let Ψ be a subroot system of Φ. Then there exists a function $p^\Psi : \mathrm{Gr}(\Psi) \to \mathbb{Z}, \alpha \mapsto p^\Psi_\alpha$, and non-negative integers n^Ψ_α for each $\alpha \in \mathrm{Gr}(\Psi)$ such that $Z_\alpha(\Psi) = p^\Psi_\alpha + n^\Psi_\alpha \mathbb{Z}$. Moreover the function p^Ψ is $\mathbb{Z}-$linear if $\mathtt{BC_r}$ is not a subroot system of $\mathrm{Gr}(\Psi)$.
(2) We have $n^\Psi_\alpha = n^\Psi_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$ with $\beta \in W_{\mathrm{Gr}(\Psi)}\alpha$.

- Let $\Psi \leq \mathrm{Gr}(\Phi)$ be a subroot system. The lift of Ψ in Φ is defined to be

$$\widehat{\Psi} := \bigcup_{\alpha \in \Psi} \{\alpha + r\delta : \text{ for all } r \text{ such that } \alpha + r\delta \in \Phi\}$$

Our work

Proposition (Dyer-Lehrer, KV)

(1) Let Φ be an irreducible affine root system and let Ψ be a subroot system of Φ. Then there exists a function $p^{\Psi} : \mathrm{Gr}(\Psi) \to \mathbb{Z}, \alpha \mapsto p^{\Psi}_{\alpha}$, and non-negative integers n^{Ψ}_{α} for each $\alpha \in \mathrm{Gr}(\Psi)$ such that $Z_{\alpha}(\Psi) = p^{\Psi}_{\alpha} + n^{\Psi}_{\alpha}\mathbb{Z}$. Moreover the function p^{Ψ} is $\mathbb{Z}-$linear if $\mathtt{BC_r}$ is not a subroot system of $\mathrm{Gr}(\Psi)$.
(2) We have $n^{\Psi}_{\alpha} = n^{\Psi}_{\beta}$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$ with $\beta \in W_{\mathrm{Gr}(\Psi)}\alpha$.

- Let $\Psi \leq \mathrm{Gr}(\Phi)$ be a subroot system. The lift of Ψ in Φ is defined to be

$$\widehat{\Psi} := \bigcup_{\alpha \in \Psi} \{\alpha + r\delta : \text{ for all } r \text{ such that } \alpha + r\delta \in \Phi\}$$

- The lift $\widehat{\Psi}$ is a closed subroot system of Φ if Ψ a closed subroot system of $\mathrm{Gr}(\Phi)$.

Proposition

Let Φ be an irreducible untwisted affine root system.

1. Suppose Ψ is a closed subroot system of Φ with an irreducible gradient subroot system $\mathrm{Gr}(\Psi)$, then $n_\alpha = n_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$. Denote this unique number by n_Ψ.
2. Suppose Ψ is a maximal closed subroot system of Φ with $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, then n_Ψ must be a prime number.

Untwisted case

- The lift $\widehat{\Psi}$ is a closed subroot system of Φ if Ψ a closed subroot system of $\mathrm{Gr}(\Phi)$.

Proposition

Let Φ be an irreducible untwisted affine root system.

1. Suppose Ψ is a closed subroot system of Φ with an irreducible gradient subroot system $\mathrm{Gr}(\Psi)$, then $n_\alpha = n_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$. Denote this unique number by n_Ψ.
2. Suppose Ψ is a maximal closed subroot system of Φ with $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, then n_Ψ must be a prime number.

Untwisted case

- The lift $\widehat{\Psi}$ is a closed subroot system of Φ if Ψ a closed subroot system of $\mathrm{Gr}(\Phi)$.

Proposition

Let Φ be an irreducible untwisted affine root system.

1. Suppose Ψ is a closed subroot system of Φ with an irreducible gradient subroot system $\mathrm{Gr}(\Psi)$, then $n_\alpha = n_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$. Denote this unique number by n_Ψ.
2. Suppose Ψ is a maximal closed subroot system of Φ with $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, then n_Ψ must be a prime number.

Untwisted case

- The lift $\widehat{\Psi}$ is a closed subroot system of Φ if Ψ a closed subroot system of $\mathrm{Gr}(\Phi)$.

Proposition

Let Φ be an irreducible untwisted affine root system.

1. Suppose Ψ is a closed subroot system of Φ with an irreducible gradient subroot system $\mathrm{Gr}(\Psi)$, then $n_\alpha = n_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$. Denote this unique number by n_Ψ.
2. Suppose Ψ is a maximal closed subroot system of Φ with $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, then n_Ψ must be a prime number.

Untwisted case

Proposition

Let Φ be an irreducible untwisted affine root system and let $\Psi \leq \Phi$ be a subroot system.

1. If $\Psi \leq \Phi$ is a closed subroot system, then $\mathrm{Gr}(\Psi) \leq \mathring{\Phi}$ is a closed subroot system.
2. If $\Psi \leq \Phi$ is a maximal closed subroot system, then either $\mathrm{Gr}(\Psi) = \mathring{\Phi}$ or $\mathrm{Gr}(\Psi) \subsetneq \mathring{\Phi}$ is a maximal closed subroot system. In particular we get $\Psi = \widehat{\mathrm{Gr}(\Psi)}$ when $\mathrm{Gr}(\Psi) \subsetneq \mathring{\Phi}$.

Untwisted case

Proposition

Let Φ be an irreducible untwisted affine root system and let $\Psi \leq \Phi$ be a subroot system.

1. If $\Psi \leq \Phi$ is a closed subroot system, then $\mathrm{Gr}(\Psi) \leq \mathring{\Phi}$ is a closed subroot system.
2. If $\Psi \leq \Phi$ is a maximal closed subroot system, then either $\mathrm{Gr}(\Psi) = \mathring{\Phi}$ or $\mathrm{Gr}(\Psi) \subsetneq \mathring{\Phi}$ is a maximal closed subroot system. In particular we get $\Psi = \widehat{\mathrm{Gr}(\Psi)}$ when $\mathrm{Gr}(\Psi) \subsetneq \mathring{\Phi}$.

Untwisted case

Proposition

Let Φ be an irreducible untwisted affine root system and let $\Psi \leq \Phi$ be a subroot system.

1. If $\Psi \leq \Phi$ is a closed subroot system, then $\mathrm{Gr}(\Psi) \leq \mathring{\Phi}$ is a closed subroot system.
2. If $\Psi \leq \Phi$ is a maximal closed subroot system, then either $\mathrm{Gr}(\Psi) = \mathring{\Phi}$ or $\mathrm{Gr}(\Psi) \subsetneq \mathring{\Phi}$ is a maximal closed subroot system. In particular we get $\Psi = \widehat{\mathrm{Gr}(\Psi)}$ when $\mathrm{Gr}(\Psi) \subsetneq \mathring{\Phi}$.

Untwisted case

Theorem (DL, FRT, KV)

Let Ψ be a maximal closed subroot system of Φ.

1. *If $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, then there exists a $\mathbb{Z}$–linear function $p : \mathrm{Gr}(\Psi) \to \mathbb{Z}$ and a prime number n_Ψ such that*

$$\Psi = \{\alpha + (p_\alpha + rn_\Psi)\delta : \alpha \in \mathrm{Gr}(\Psi), r \in \mathbb{Z}\}.$$

2. *If $\mathrm{Gr}(\Psi) \subsetneq \mathring{\Phi}$ is a maximal closed subroot system, then*

$$\Psi = \{\alpha + r\delta : \alpha \in \mathrm{Gr}(\Psi), r \in \mathbb{Z}\}.$$

Converse is also true in both cases.

Untwisted case

Theorem (DL, FRT, KV)

Let Ψ be a maximal closed subroot system of Φ.

1. *If $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, then there exists a $\mathbb{Z}$–linear function $p : \mathrm{Gr}(\Psi) \to \mathbb{Z}$ and a prime number n_Ψ such that*

$$\Psi = \{\alpha + (p_\alpha + rn_\Psi)\delta : \alpha \in \mathrm{Gr}(\Psi), r \in \mathbb{Z}\}.$$

2. *If $\mathrm{Gr}(\Psi) \subsetneq \mathring{\Phi}$ is a maximal closed subroot system, then*

$$\Psi = \{\alpha + r\delta : \alpha \in \mathrm{Gr}(\Psi), r \in \mathbb{Z}\}.$$

Converse is also true in both cases.

Untwisted case

Theorem (DL, FRT, KV)

Let Ψ be a maximal closed subroot system of Φ.

1. *If $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, then there exists a $\mathbb{Z}$–linear function $p : \mathrm{Gr}(\Psi) \to \mathbb{Z}$ and a prime number n_Ψ such that*

$$\Psi = \{\alpha + (p_\alpha + rn_\Psi)\delta : \alpha \in \mathrm{Gr}(\Psi), r \in \mathbb{Z}\}.$$

2. *If $\mathrm{Gr}(\Psi) \subsetneq \mathring{\Phi}$ is a maximal closed subroot system, then*

$$\Psi = \{\alpha + r\delta : \alpha \in \mathrm{Gr}(\Psi), r \in \mathbb{Z}\}.$$

Converse is also true in both cases.

Types of maximal closed subroot systems of irreducible finite root systems:

Type	Reducible	Irreducible
A_n	$A_r \oplus A_{n-r-1}$ $(1 \leq r \leq n-1)$	A_{n-1}, A_n
B_n	$B_r \oplus D_{n-r}$ $(1 \leq r \leq n-2)$	B_{n-1}, D_n
C_n	$C_r \oplus C_{n-r}$ $(1 \leq r \leq n-1)$	A_{n-1}
D_n	$D_r \oplus D_{n-r}$ $(2 \leq r \leq n-2)$	A_{n-1}, D_{n-1}, D_n
E_6	$A_5 \oplus A_1, A_2 \oplus A_2 \oplus A_2$	D_5, E_6
E_7	$A_5 \oplus A_2, A_1 \oplus D_6$	E_6, A_7, E_7
E_8	$A_1 \oplus E_7, E_6 \oplus A_2, A_4 \oplus A_4$	D_8, A_8, E_8
F_4	$A_2 \oplus A_2, C_3 \oplus A_1$	B_4
G_2	$A_1 \oplus A_1$	A_2

Types of maximal closed subroot systems of irreducible untwisted affine root systems:

Type	Reducible	Irreducible
$A_n^{(1)}$	$A_r^{(1)} \oplus A_{n-r-1}^{(1)} \ (0 \leq r \leq n-1)$	$A_n^{(1)}$
$B_n^{(1)}$	$B_r^{(1)} \oplus D_{n-r}^{(1)} \quad (1 \leq r \leq n-2)$	$B_{n-1}^{(1)}, D_n^{(1)}, B_n^{(1)}$
$C_n^{(1)}$	$C_r^{(1)} \oplus C_{n-r}^{(1)} \quad (1 \leq r \leq n-1)$	$A_{n-1}^{(1)}, C_n^{(1)}$
$D_n^{(1)}$	$D_r^{(1)} \oplus D_{n-r}^{(1)} \quad (2 \leq r \leq n-2)$	$A_{n-1}^{(1)}, D_{n-1}^{(1)}, D_n^{(1)}$
$E_6^{(1)}$	$A_5^{(1)} \oplus A_1^{(1)}, A_2^{(1)} \oplus A_2^{(1)} \oplus A_2^{(1)}$	$D_5^{(1)}, E_6^{(1)}$
$E_7^{(1)}$	$A_5^{(1)} \oplus A_2^{(1)}, A_1^{(1)} \oplus D_6^{(1)}$	$E_6^{(1)}, A_7^{(1)}, E_7^{(1)}$
$E_8^{(1)}$	$A_1^{(1)} \oplus E_7^{(1)}, E_6^{(1)} \oplus A_2^{(1)}, A_4^{(1)} \oplus A_4^{(1)}$	$D_8^{(1)}, A_8^{(1)}, E_8^{(1)}$
$F_4^{(1)}$	$A_2^{(1)} \oplus A_2^{(1)}, A_1^{(1)} \oplus C_3^{(1)}$	$B_4^{(1)}, F_4^{(1)}$
$G_2^{(1)}$	$A_1^{(1)} \oplus A_1^{(1)}$	$A_2^{(1)}, G_2^{(1)}$

Twisted case

Definition

A subroot system $\mathring{\Psi}$ of $\mathring{\Phi}$ is said to be *semi-closed* if

1. $\mathring{\Psi}$ is not closed in $\mathring{\Phi}$ and
2. if $\alpha, \beta \in \mathring{\Psi}$ such that $\alpha + \beta \in \mathring{\Phi} \backslash \mathring{\Psi}$, then α and β must be short roots/intermediate roots and $\alpha + \beta$ must be a long root.

Proposition

Let Φ be an irreducible twisted affine root system. If $\Psi \leq \Phi$ is a closed subroot system, then we have either

1. $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Phi)$ or $\mathrm{Gr}(\Psi)$ is a proper closed subroot system of $\mathrm{Gr}(\Phi)$ or
2. $\mathrm{Gr}(\Psi)$ is a proper semi-closed subroot system of $\mathrm{Gr}(\Phi)$.

Twisted case

Definition

A subroot system $\mathring{\Psi}$ of $\mathring{\Phi}$ is said to be *semi-closed* if

1. $\mathring{\Psi}$ is not closed in $\mathring{\Phi}$ and
2. if $\alpha, \beta \in \mathring{\Psi}$ such that $\alpha + \beta \in \mathring{\Phi} \backslash \mathring{\Psi}$, then α and β must be short roots/intermediate roots and $\alpha + \beta$ must be a long root.

Proposition

Let Φ be an irreducible twisted affine root system. If $\Psi \leq \Phi$ is a closed subroot system, then we have either

1. $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Phi)$ or $\mathrm{Gr}(\Psi)$ is a proper closed subroot system of $\mathrm{Gr}(\Phi)$ or
2. $\mathrm{Gr}(\Psi)$ is a proper semi-closed subroot system of $\mathrm{Gr}(\Phi)$.

Twisted case

Definition

A subroot system $\mathring{\Psi}$ of $\mathring{\Phi}$ is said to be *semi-closed* if

1. $\mathring{\Psi}$ is not closed in $\mathring{\Phi}$ and
2. if $\alpha, \beta \in \mathring{\Psi}$ such that $\alpha + \beta \in \mathring{\Phi} \backslash \mathring{\Psi}$, then α and β must be short roots/intermediate roots and $\alpha + \beta$ must be a long root.

Proposition

Let Φ be an irreducible twisted affine root system. If $\Psi \leq \Phi$ is a closed subroot system, then we have either

1. $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Phi)$ or $\mathrm{Gr}(\Psi)$ is a proper closed subroot system of $\mathrm{Gr}(\Phi)$ or
2. $\mathrm{Gr}(\Psi)$ is a proper semi-closed subroot system of $\mathrm{Gr}(\Phi)$.

Twisted case

Definition

A subroot system $\mathring{\Psi}$ of $\mathring{\Phi}$ is said to be *semi-closed* if

1. $\mathring{\Psi}$ is not closed in $\mathring{\Phi}$ and
2. if $\alpha, \beta \in \mathring{\Psi}$ such that $\alpha + \beta \in \mathring{\Phi} \backslash \mathring{\Psi}$, then α and β must be short roots/intermediate roots and $\alpha + \beta$ must be a long root.

Proposition

Let Φ be an irreducible twisted affine root system. If $\Psi \leq \Phi$ is a closed subroot system, then we have either

1. $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Phi)$ or $\mathrm{Gr}(\Psi)$ is a proper closed subroot system of $\mathrm{Gr}(\Phi)$ or
2. $\mathrm{Gr}(\Psi)$ is a proper semi-closed subroot system of $\mathrm{Gr}(\Phi)$.

Twisted case

Definition

A subroot system $\mathring{\Psi}$ of $\mathring{\Phi}$ is said to be *semi-closed* if

1. $\mathring{\Psi}$ is not closed in $\mathring{\Phi}$ and
2. if $\alpha, \beta \in \mathring{\Psi}$ such that $\alpha + \beta \in \mathring{\Phi} \backslash \mathring{\Psi}$, then α and β must be short roots/intermediate roots and $\alpha + \beta$ must be a long root.

Proposition

Let Φ be an irreducible twisted affine root system. If $\Psi \leq \Phi$ is a closed subroot system, then we have either

1. $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Phi)$ or $\mathrm{Gr}(\Psi)$ is a proper closed subroot system of $\mathrm{Gr}(\Phi)$ or
2. $\mathrm{Gr}(\Psi)$ is a proper semi-closed subroot system of $\mathrm{Gr}(\Phi)$.

Twisted case

Definition

A subroot system $\mathring{\Psi}$ of $\mathring{\Phi}$ is said to be *semi-closed* if

1. $\mathring{\Psi}$ is not closed in $\mathring{\Phi}$ and
2. if $\alpha, \beta \in \mathring{\Psi}$ such that $\alpha + \beta \in \mathring{\Phi} \backslash \mathring{\Psi}$, then α and β must be short roots/intermediate roots and $\alpha + \beta$ must be a long root.

Proposition

Let Φ be an irreducible twisted affine root system. If $\Psi \leq \Phi$ is a closed subroot system, then we have either

1. $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Phi)$ or $\mathrm{Gr}(\Psi)$ is a proper closed subroot system of $\mathrm{Gr}(\Phi)$ or
2. $\mathrm{Gr}(\Psi)$ is a proper semi-closed subroot system of $\mathrm{Gr}(\Phi)$.

Twisted case: not of type $A_{2n}^{(2)}$

Proposition

Let Φ be an irreducible twisted affine root system not of type $A_{2n}^{(2)}$ and let $\Psi \leq \Phi$ be a closed subroot system with an irreducible gradient subroot system $\mathrm{Gr}(\Psi)$.

1. Suppose $\mathrm{Gr}(\Psi)$ is simply laced, then we get $n_\alpha = n_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$. Denote this unique number by n_Ψ.
2. Suppose $\mathrm{Gr}(\Psi)$ is non simply-laced, then we get $n_\ell = n_s$ if $m | n_s$ and we get $n_\ell = mn_s$ if $m \nmid n_s$.
3. Suppose $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, then n_s is a prime number.

Now let Φ be of type $A_{2n}^{(2)}$ and let $\Psi \leq \Phi$ be a closed subroot system with an irreducible gradient subroot system $\mathrm{Gr}(\Psi)$.

Proposition

Let Φ be an irreducible twisted affine root system not of type $A_{2n}^{(2)}$ and let $\Psi \leq \Phi$ be a closed subroot system with an irreducible gradient subroot system $\mathrm{Gr}(\Psi)$.

1. Suppose $\mathrm{Gr}(\Psi)$ is simply laced, then we get $n_\alpha = n_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$. Denote this unique number by n_Ψ.
2. Suppose $\mathrm{Gr}(\Psi)$ is non simply-laced, then we get $n_\ell = n_s$ if $m|n_s$ and we get $n_\ell = mn_s$ if $m \nmid n_s$.
3. Suppose $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, then n_s is a prime number.

Now let Φ be of type $A_{2n}^{(2)}$ and let $\Psi \leq \Phi$ be a closed subroot system with an irreducible gradient subroot system $\mathrm{Gr}(\Psi)$.

Twisted case: not of type $A_{2n}^{(2)}$

Proposition

Let Φ be an irreducible twisted affine root system not of type $A_{2n}^{(2)}$ and let $\Psi \leq \Phi$ be a closed subroot system with an irreducible gradient subroot system $\mathrm{Gr}(\Psi)$.

1. Suppose $\mathrm{Gr}(\Psi)$ is simply laced, then we get $n_\alpha = n_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$. Denote this unique number by n_Ψ.
2. Suppose $\mathrm{Gr}(\Psi)$ is non simply-laced, then we get $n_\ell = n_s$ if $m | n_s$ and we get $n_\ell = mn_s$ if $m \not| n_s$.
3. Suppose $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, then n_s is a prime number.

Now let Φ be of type $A_{2n}^{(2)}$ and let $\Psi \leq \Phi$ be a closed subroot system with an irreducible gradient subroot system $\mathrm{Gr}(\Psi)$.

Twisted case: not of type $A_{2n}^{(2)}$

Proposition

Let Φ be an irreducible twisted affine root system not of type $A_{2n}^{(2)}$ and let $\Psi \le \Phi$ be a closed subroot system with an irreducible gradient subroot system $\mathrm{Gr}(\Psi)$.

1. Suppose $\mathrm{Gr}(\Psi)$ is simply laced, then we get $n_\alpha = n_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$. Denote this unique number by n_Ψ.
2. Suppose $\mathrm{Gr}(\Psi)$ is non simply-laced, then we get $n_\ell = n_s$ if $m | n_s$ and we get $n_\ell = mn_s$ if $m \not| n_s$.
3. Suppose $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, then n_s is a prime number.

Now let Φ be of type $A_{2n}^{(2)}$ and let $\Psi \le \Phi$ be a closed subroot system with an irreducible gradient subroot system $\mathrm{Gr}(\Psi)$.

Twisted case: $A_{2n}^{(2)}$

Proposition

1. Suppose $\mathrm{Gr}(\Psi)$ is simply laced, then we get $n_\alpha = n_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$.
2. Suppose $\mathrm{Gr}(\Psi)$ is non-simply laced and does not contain any short root, then we get $n_\ell = n_{\mathrm{im}}$ if $2 | n_{\mathrm{im}}$ and we get $n_\ell = 2n_{\mathrm{im}}$ if $2 \not| n_{\mathrm{im}}$.
3. Suppose $\mathrm{Gr}(\Psi)$ is non-simply laced and does not contain any long root, then we get $n_s = n_{\mathrm{im}}$.
4. Suppose $\mathrm{Gr}(\Psi)$ containing short, intermediate and long roots, then $n_s = n_{\mathrm{im}}$, $n_\ell = 2n_s$ and n_s is an odd number.
5. Suppose Ψ is a maximal closed subroot system of Φ with $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Phi)$, then n_s must be a prime number.

Twisted case: $\mathtt{A}_{2n}^{(2)}$

Proposition

1. Suppose $\mathrm{Gr}(\Psi)$ is simply laced, then we get $n_\alpha = n_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$.
2. Suppose $\mathrm{Gr}(\Psi)$ is non-simply laced and does not contain any short root, then we get $n_\ell = n_{\mathrm{im}}$ if $2|n_{\mathrm{im}}$ and we get $n_\ell = 2n_{\mathrm{im}}$ if $2 \not| n_{\mathrm{im}}$.
3. Suppose $\mathrm{Gr}(\Psi)$ is non-simply laced and does not contain any long root, then we get $n_s = n_{\mathrm{im}}$.
4. Suppose $\mathrm{Gr}(\Psi)$ containing short, intermediate and long roots, then $n_s = n_{\mathrm{im}}$, $n_\ell = 2n_s$ and n_s is an odd number.
5. Suppose Ψ is a maximal closed subroot system of Φ with $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Phi)$, then n_s must be a prime number.

Twisted case: $\mathtt{A}_{2n}^{(2)}$

Proposition

1. Suppose $\mathrm{Gr}(\Psi)$ is simply laced, then we get $n_\alpha = n_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$.
2. Suppose $\mathrm{Gr}(\Psi)$ is non-simply laced and does not contain any short root, then we get $n_\ell = n_{\mathrm{im}}$ if $2 | n_{\mathrm{im}}$ and we get $n_\ell = 2n_{\mathrm{im}}$ if $2 \not| n_{\mathrm{im}}$.
3. Suppose $\mathrm{Gr}(\Psi)$ is non-simply laced and does not contain any long root, then we get $n_s = n_{\mathrm{im}}$.
4. Suppose $\mathrm{Gr}(\Psi)$ containing short, intermediate and long roots, then $n_s = n_{\mathrm{im}}$, $n_\ell = 2n_s$ and n_s is an odd number.
5. Suppose Ψ is a maximal closed subroot system of Φ with $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Phi)$, then n_s must be a prime number.

Twisted case: $\mathtt{A}_{2n}^{(2)}$

Proposition

1. Suppose $\mathrm{Gr}(\Psi)$ is simply laced, then we get $n_\alpha = n_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$.
2. Suppose $\mathrm{Gr}(\Psi)$ is non-simply laced and does not contain any short root, then we get $n_\ell = n_{\mathrm{im}}$ if $2 | n_{\mathrm{im}}$ and we get $n_\ell = 2n_{\mathrm{im}}$ if $2 \not| n_{\mathrm{im}}$.
3. Suppose $\mathrm{Gr}(\Psi)$ is non-simply laced and does not contain any long root, then we get $n_s = n_{\mathrm{im}}$.
4. Suppose $\mathrm{Gr}(\Psi)$ containing short, intermediate and long roots, then $n_s = n_{\mathrm{im}}$, $n_\ell = 2n_s$ and n_s is an odd number.
5. Suppose Ψ is a maximal closed subroot system of Φ with $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Phi)$, then n_s must be a prime number.

Twisted case: $\mathtt{A}_{2n}^{(2)}$

Proposition

1. Suppose $\mathrm{Gr}(\Psi)$ is simply laced, then we get $n_\alpha = n_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$.
2. Suppose $\mathrm{Gr}(\Psi)$ is non-simply laced and does not contain any short root, then we get $n_\ell = n_{\mathrm{im}}$ if $2 | n_{\mathrm{im}}$ and we get $n_\ell = 2n_{\mathrm{im}}$ if $2 \not| n_{\mathrm{im}}$.
3. Suppose $\mathrm{Gr}(\Psi)$ is non-simply laced and does not contain any long root, then we get $n_s = n_{\mathrm{im}}$.
4. Suppose $\mathrm{Gr}(\Psi)$ containing short, intermediate and long roots, then $n_s = n_{\mathrm{im}}$, $n_\ell = 2n_s$ and n_s is an odd number.
5. Suppose Ψ is a maximal closed subroot system of Φ with $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Phi)$, then n_s must be a prime number.

Twisted case: $A_{2n}^{(2)}$

Proposition

1. Suppose $\mathrm{Gr}(\Psi)$ is simply laced, then we get $n_\alpha = n_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$.
2. Suppose $\mathrm{Gr}(\Psi)$ is non-simply laced and does not contain any short root, then we get $n_\ell = n_{\mathrm{im}}$ if $2 | n_{\mathrm{im}}$ and we get $n_\ell = 2n_{\mathrm{im}}$ if $2 \not| n_{\mathrm{im}}$.
3. Suppose $\mathrm{Gr}(\Psi)$ is non-simply laced and does not contain any long root, then we get $n_s = n_{\mathrm{im}}$.
4. Suppose $\mathrm{Gr}(\Psi)$ containing short, intermediate and long roots, then $n_s = n_{\mathrm{im}}$, $n_\ell = 2n_s$ and n_s is an odd number.
5. Suppose Ψ is a maximal closed subroot system of Φ with $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Phi)$, then n_s must be a prime number.

Twisted case: not of type $A_{2n}^{(2)}$

Theorem

Let Φ be an irreducible twisted affine root system which is not of type $A_{2n}^{(2)}$ and let Ψ be a maximal closed subroot system of Φ.

1. *If $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, then there exists a $\mathbb{Z}$–linear function $p : \mathrm{Gr}(\Psi) \to \mathbb{Z}$ and a prime number n_s, $p_\alpha \in m\mathbb{Z}$ for long roots α such that*
   - $\Psi = \{\alpha + (p_\alpha + rn_\Psi)\delta, \beta + (p_\beta + mrn_\Psi)\delta : \alpha \in \mathring{\Phi}_s, \beta \in \mathring{\Phi}_\ell, r \in \mathbb{Z}\}$ *if* $m \neq n_\Psi$
   - $\Psi = \{\alpha + (p_\alpha + rn_\Psi)\delta : \alpha \in \mathring{\Phi}, \ r \in \mathbb{Z}\}$ *if* $m = n_\Psi$.
2. *If $\mathrm{Gr}(\Psi) \subsetneq \mathring{\Phi}$ is a proper closed subroot system, then $\mathrm{Gr}(\Psi) < \mathring{\Phi}$ is a maximal closed subroot system such that it contains at least one short root and in this case $\Psi = \widehat{\mathrm{Gr}(\Psi)}$.*

Twisted case: not of type $A_{2n}^{(2)}$

Theorem

Let Φ be an irreducible twisted affine root system which is not of type $A_{2n}^{(2)}$ and let Ψ be a maximal closed subroot system of Φ.

1. *If $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, then there exists a $\mathbb{Z}$–linear function $p : \mathrm{Gr}(\Psi) \to \mathbb{Z}$ and a prime number n_s, $p_\alpha \in m\mathbb{Z}$ for long roots α such that*
   - $\Psi = \{\alpha + (p_\alpha + rn_\Psi)\delta, \beta + (p_\beta + mrn_\Psi)\delta : \alpha \in \mathring{\Phi}_s, \beta \in \mathring{\Phi}_\ell, r \in \mathbb{Z}\}$ *if* $m \neq n_\Psi$
   - $\Psi = \{\alpha + (p_\alpha + rn_\Psi)\delta : \alpha \in \mathring{\Phi},\ r \in \mathbb{Z}\}$ *if* $m = n_\Psi$.
2. *If $\mathrm{Gr}(\Psi) \subsetneq \mathring{\Phi}$ is a proper closed subroot system, then $\mathrm{Gr}(\Psi) < \mathring{\Phi}$ is a maximal closed subroot system such that it contains at least one short root and in this case $\Psi = \widehat{\mathrm{Gr}(\Psi)}$.*

Twisted case: not of type $\mathtt{A}_{2n}^{(2)}$

Theorem

Let Φ be an irreducible twisted affine root system which is not of type $\mathtt{A}_{2n}^{(2)}$ and let Ψ be a maximal closed subroot system of Φ.

1. *If $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, then there exists a $\mathbb{Z}$–linear function $p : \mathrm{Gr}(\Psi) \to \mathbb{Z}$ and a prime number n_s, $p_\alpha \in m\mathbb{Z}$ for long roots α such that*
   - $\Psi = \{\alpha + (p_\alpha + rn_\Psi)\delta, \beta + (p_\beta + mrn_\Psi)\delta : \alpha \in \mathring{\Phi}_s, \beta \in \mathring{\Phi}_\ell, r \in \mathbb{Z}\}$ *if* $m \neq n_\Psi$
   - $\Psi = \{\alpha + (p_\alpha + rn_\Psi)\delta : \alpha \in \mathring{\Phi}, \ r \in \mathbb{Z}\}$ *if* $m = n_\Psi$.
2. *If $\mathrm{Gr}(\Psi) \subsetneq \mathring{\Phi}$ is a proper closed subroot system, then $\mathrm{Gr}(\Psi) < \mathring{\Phi}$ is a maximal closed subroot system such that it contains at least one short root and in this case $\Psi = \widehat{\mathrm{Gr}(\Psi)}$.*

Twisted case: not of type $\mathtt{A}_{2n}^{(2)}$

Theorem

Let Φ be an irreducible twisted affine root system which is not of type $\mathtt{A}_{2n}^{(2)}$ and let Ψ be a maximal closed subroot system of Φ.

1. *If $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, then there exists a $\mathbb{Z}$–linear function $p : \mathrm{Gr}(\Psi) \to \mathbb{Z}$ and a prime number n_s, $p_\alpha \in m\mathbb{Z}$ for long roots α such that*
   - $\Psi = \{\alpha + (p_\alpha + rn_\Psi)\delta, \beta + (p_\beta + mrn_\Psi)\delta : \alpha \in \mathring{\Phi}_s, \beta \in \mathring{\Phi}_\ell, r \in \mathbb{Z}\}$ *if* $m \neq n_\Psi$
   - $\Psi = \{\alpha + (p_\alpha + rn_\Psi)\delta : \alpha \in \mathring{\Phi},\ r \in \mathbb{Z}\}$ *if* $m = n_\Psi$.
2. *If $\mathrm{Gr}(\Psi) \subsetneq \mathring{\Phi}$ is a proper closed subroot system, then $\mathrm{Gr}(\Psi) < \mathring{\Phi}$ is a maximal closed subroot system such that it contains at least one short root and in this case $\Psi = \widehat{\mathrm{Gr}(\Psi)}$.*

Twisted case: not of type $\mathtt{A}_{2n}^{(2)}$

Theorem

Let Φ be an irreducible twisted affine root system which is not of type $\mathtt{A}_{2n}^{(2)}$ and let Ψ be a maximal closed subroot system of Φ.

1. *If $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, then there exists a $\mathbb{Z}$–linear function $p : \mathrm{Gr}(\Psi) \to \mathbb{Z}$ and a prime number n_s, $p_\alpha \in m\mathbb{Z}$ for long roots α such that*
   - $\Psi = \{\alpha + (p_\alpha + rn_\Psi)\delta, \beta + (p_\beta + mrn_\Psi)\delta : \alpha \in \mathring{\Phi}_s, \beta \in \mathring{\Phi}_\ell, r \in \mathbb{Z}\}$ *if* $m \neq n_\Psi$
   - $\Psi = \{\alpha + (p_\alpha + rn_\Psi)\delta : \alpha \in \mathring{\Phi},\ r \in \mathbb{Z}\}$ *if* $m = n_\Psi$.
2. *If $\mathrm{Gr}(\Psi) \subsetneq \mathring{\Phi}$ is a proper closed subroot system, then $\mathrm{Gr}(\Psi) < \mathring{\Phi}$ is a maximal closed subroot system such that it contains at least one short root and in this case $\Psi = \widehat{\mathrm{Gr}(\Psi)}$.*

Twisted case (semi-closed example): $\mathrm{D}^{(2)}_{n+1}$

Definition

For a subset $I \subseteq I_n$, we define
$\Psi_I(\mathrm{D}^{(2)}_{n+1}) = \{\pm\epsilon_s + 2r\delta : s \in I, r \in \mathbb{Z}\} \cup \{\pm\epsilon_s + (2r+1)\delta : s \notin I, r \in \mathbb{Z}\} \cup \{\pm\epsilon_s \pm \epsilon_t + 2r\delta : s \neq t, \ \ s, t \in I \text{ or } s, t \notin I, \ r \in \mathbb{Z}\}.$

Proposition

Suppose Φ is of type $\mathrm{D}^{(2)}_{n+1}$ and $\Psi \leq \Phi$ is a maximal closed subroot system with proper semi-closed gradient subroot system $\mathrm{Gr}(\Psi) < \mathring{\Phi}$, then there exist a set $I \subsetneq I_n$ such that $\Psi = \Psi_I(\mathrm{D}^{(2)}_{n+1})$.

Twisted case (semi-closed example): $\mathrm{D}_{n+1}^{(2)}$

Definition

For a subset $I \subseteq I_n$, we define $\Psi_I(\mathrm{D}_{n+1}^{(2)}) = \{\pm\epsilon_s + 2r\delta : s \in I, r \in \mathbb{Z}\} \cup \{\pm\epsilon_s + (2r+1)\delta : s \notin I, r \in \mathbb{Z}\} \cup \{\pm\epsilon_s \pm \epsilon_t + 2r\delta : s \neq t, \ \ s, t \in I \text{ or } s, t \notin I, \ r \in \mathbb{Z}\}$.

Proposition

Suppose Φ is of type $\mathrm{D}_{n+1}^{(2)}$ and $\Psi \leq \Phi$ is a maximal closed subroot system with proper semi-closed gradient subroot system $\mathrm{Gr}(\Psi) < \mathring{\Phi}$, then there exist a set $I \subsetneq I_n$ such that $\Psi = \Psi_I(\mathrm{D}_{n+1}^{(2)})$.

Table: Types of maximal subroot system of irreducible twisted affine root systems

Type	With closed gradient
$A_2^{(2)}$	$A_2^{(2)}$
$A_{2n}^{(2)}$	$A_{2r}^{(2)} \oplus A_{2n-2r-1}^{(2)}$ $(1 \leq r \leq n-1)$, $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$
$D_{n+1}^{(2)}$	$D_{r+1}^{(2)} \oplus D_{n-r}^{(1)}$ $(1 \leq r \leq n-2)$, $B_n^{(1)}$, $D_{n+1}^{(2)}$, $D_n^{(2)}$
$A_{2n-1}^{(2)}$	$A_{2r-1}^{(2)} \oplus A_{2n-2r-1}^{(2)}$ $(1 \leq r \leq n-1)$, $A_{2n-1}^{(2)}$, $C_n^{(1)}$, $A_{n-1}^{(1)}$
$E_6^{(2)}$	$A_1^{(1)} \oplus A_5^{(2)}$, $A_2^{(1)} \oplus A_2^{(1)}$ $E_6^{(2)}$, $F_4^{(1)}$, $D_5^{(2)}$
$D_4^{(3)}$	$A_1^{(1)} \oplus A_1^{(1)}$, $D_4^{(3)}$, $G_2^{(1)}$, $A_2^{(1)}$

Table: Types of maximal subroot system of irreducible twisted affine root systems

Type	With semi-closed gradient
$A_2^{(2)}$	$A_1^{(1)}$
$A_{2n}^{(2)}$	$B_n^{(1)}$
$D_{n+1}^{(2)}$	$B_r^{(1)} \oplus B_{n-r}^{(1)} \quad (2 \leq r \leq n-2)$
$A_{2n-1}^{(2)}$	$D_n^{(1)}$
$E_6^{(2)}$	$C_4^{(1)}$
$D_4^{(3)}$	$A_2^{(1)}$

Future work/Problems

- Affine reflection systems are natural generalization of real affine root systems. The abelian group appear in affine reflection systems are not simple enough to handle. Me and Deniz have classified all maximal closed subroot systems of affine reflection systems.
- Classify all semi-closed subroot systems of finite root systems.
- Classify all subalgebras of affine Lie algebras which is of Kac-Moody type.
- Classify all regular subalgebras of affine Lie super algebras.
- Classify all maximal closed subroot systems of Extended Affine Root supersystems.

Future work/Problems

- Affine reflection systems are natural generalization of real affine root systems. The abelian group appear in affine reflection systems are not simple enough to handle. Me and Deniz have classified all maximal closed subroot systems of affine reflection systems.
- Classify all semi-closed subroot systems of finite root systems.
- Classify all subalgebras of affine Lie algebras which is of Kac-Moody type.
- Classify all regular subalgebras of affine Lie super algebras.
- Classify all maximal closed subroot systems of Extended Affine Root supersystems.

Future work/Problems

- Affine reflection systems are natural generalization of real affine root systems. The abelian group appear in affine reflection systems are not simple enough to handle. Me and Deniz have classified all maximal closed subroot systems of affine reflection systems.
- Classify all semi-closed subroot systems of finite root systems.
- Classify all subalgebras of affine Lie algebras which is of Kac-Moody type.
- Classify all regular subalgebras of affine Lie super algebras.
- Classify all maximal closed subroot systems of Extended Affine Root supersystems.

Future work/Problems

- Affine reflection systems are natural generalization of real affine root systems. The abelian group appear in affine reflection systems are not simple enough to handle. Me and Deniz have classified all maximal closed subroot systems of affine reflection systems.
- Classify all semi-closed subroot systems of finite root systems.
- Classify all subalgebras of affine Lie algebras which is of Kac-Moody type.
- Classify all regular subalgebras of affine Lie super algebras.
- Classify all maximal closed subroot systems of Extended Affine Root supersystems.

Future work/Problems

- Affine reflection systems are natural generalization of real affine root systems. The abelian group appear in affine reflection systems are not simple enough to handle. Me and Deniz have classified all maximal closed subroot systems of affine reflection systems.
- Classify all semi-closed subroot systems of finite root systems.
- Classify all subalgebras of affine Lie algebras which is of Kac-Moody type.
- Classify all regular subalgebras of affine Lie super algebras.
- Classify all maximal closed subroot systems of Extended Affine Root supersystems.

References

- A. Borel and J. De Siebenthal. Les sous-groupes fermÃ©s de rang maximum des groupes de Lie clos. Comment. Math. Helv., 23:200â221, 1949.
- M. J. Dyer and G. I. Lehrer. Reflection subgroups of finite and affine Weyl groups. Trans. Amer. Math. Soc., 363(11):5971â6005, 2011.
- M. J. Dyer and G. I. Lehrer. Root subsystems of loop extensions. Transform. Groups, 16(3):767â781, 2011.
- E. B. Dynkin. Regular semisimple subalgebras of semisimple Lie algebras. Doklady Akad. Nauk SSSR (N.S.), 73:877â880, 1950.

References

- A. Borel and J. De Siebenthal. Les sous-groupes fermÃ©s de rang maximum des groupes de Lie clos. Comment. Math. Helv., 23:200â221, 1949.
- M. J. Dyer and G. I. Lehrer. Reflection subgroups of finite and affine Weyl groups. Trans. Amer. Math. Soc., 363(11):5971â6005, 2011.
- M. J. Dyer and G. I. Lehrer. Root subsystems of loop extensions. Transform. Groups, 16(3):767â781, 2011.
- E. B. Dynkin. Regular semisimple subalgebras of semisimple Lie algebras. Doklady Akad. Nauk SSSR (N.S.), 73:877â880, 1950.

References

- A. Borel and J. De Siebenthal. Les sous-groupes fermÃ©s de rang maximum des groupes de Lie clos. Comment. Math. Helv., 23:200â221, 1949.
- M. J. Dyer and G. I. Lehrer. Reflection subgroups of finite and affine Weyl groups. Trans. Amer. Math. Soc., 363(11):5971â6005, 2011.
- M. J. Dyer and G. I. Lehrer. Root subsystems of loop extensions. Transform. Groups, 16(3):767â781, 2011.
- E. B. Dynkin. Regular semisimple subalgebras of semisimple Lie algebras. Doklady Akad. Nauk SSSR (N.S.), 73:877â880, 1950.

References

- A. Borel and J. De Siebenthal. Les sous-groupes fermÃ©s de rang maximum des groupes de Lie clos. Comment. Math. Helv., 23:200â221, 1949.
- M. J. Dyer and G. I. Lehrer. Reflection subgroups of finite and affine Weyl groups. Trans. Amer. Math. Soc., 363(11):5971â6005, 2011.
- M. J. Dyer and G. I. Lehrer. Root subsystems of loop extensions. Transform. Groups, 16(3):767â781, 2011.
- E. B. Dynkin. Regular semisimple subalgebras of semisimple Lie algebras. Doklady Akad. Nauk SSSR (N.S.), 73:877â880, 1950.

References

- Anna Felikson, Alexander Retakh, and Pavel Tumarkin. Regular subalgebras of affine Kac-Moody algebras. J. Phys. A, 41(36):365204, 16, 2008.
- Anna Felikson and Pavel Tumarkin. Hyperbolic subalgebras of hyperbolic KacâMoody algebras. Transformation Groups. Vol 17, Issue 1, pp 87â122, 2012.
- Krishanu Roy, R. Venkatesh, Maximal closed subroot systems of affine root systems. arXiv:1707.07981.
- Deniz Kus, R. Venkatesh, Borel-de Siebenthal theory for affine reflection systems. In preparation.

References

- Anna Felikson, Alexander Retakh, and Pavel Tumarkin. Regular subalgebras of affine Kac-Moody algebras. J. Phys. A, 41(36):365204, 16, 2008.
- Anna Felikson and Pavel Tumarkin. Hyperbolic subalgebras of hyperbolic KacâMoody algebras. Transformation Groups. Vol 17, Issue 1, pp 87â122, 2012.
- Krishanu Roy, R. Venkatesh, Maximal closed subroot systems of affine root systems. arXiv:1707.07981.
- Deniz Kus, R. Venkatesh, Borel-de Siebenthal theory for affine reflection systems. In preparation.

References

- Anna Felikson, Alexander Retakh, and Pavel Tumarkin. Regular subalgebras of affine Kac-Moody algebras. J. Phys. A, 41(36):365204, 16, 2008.
- Anna Felikson and Pavel Tumarkin. Hyperbolic subalgebras of hyperbolic KacâMoody algebras. Transformation Groups. Vol 17, Issue 1, pp 87â122, 2012.
- Krishanu Roy, R. Venkatesh, Maximal closed subroot systems of affine root systems. arXiv:1707.07981.
- Deniz Kus, R. Venkatesh, Borel-de Siebenthal theory for affine reflection systems. In preparation.

References

- Anna Felikson, Alexander Retakh, and Pavel Tumarkin. Regular subalgebras of affine Kac-Moody algebras. J. Phys. A, 41(36):365204, 16, 2008.
- Anna Felikson and Pavel Tumarkin. Hyperbolic subalgebras of hyperbolic KacâMoody algebras. Transformation Groups. Vol 17, Issue 1, pp 87â122, 2012.
- Krishanu Roy, R. Venkatesh, Maximal closed subroot systems of affine root systems. arXiv:1707.07981.
- Deniz Kus, R. Venkatesh, Borel-de Siebenthal theory for affine reflection systems. In preparation.

Thank you