Integrable modules for Lie tori

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- Let $m = (m_1, \cdots, m_n)$, $k = (k_1, \cdots, k_n)$ and $l = (l_1, \cdots, l_n)$ are vectors in $\mathbb{Z}^n$. Let $\Gamma = m_1\mathbb{Z} \oplus \cdots \oplus m_n\mathbb{Z}$ and $\Gamma_0 = m_0\mathbb{Z}$. Let $\Lambda = \mathbb{Z}^n/\Gamma$ and $\Lambda_0 = \mathbb{Z}/\Gamma_0$. Let $\overline{k}, \overline{l}$ denote the images in Λ. For any integers k_0 and l_0, let $\overline{k}_0$ and $\overline{l}_0$ denote images in Λ_0.

Some notations

- Let

$$\begin{array}{lcl} A & = & \mathbb{C}[t_0^{\pm 1}, \cdots, t_n^{\pm 1}], \\ A_n & = & \mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}], \\ A(m) & = & \mathbb{C}[t_1^{\pm m_1}, \cdots, t_n^{\pm m_n}], \text{ and} \\ A(m_0, m) & = & \mathbb{C}[t_0^{\pm m_0}, \cdots, t_n^{\pm m_n}] \end{array}$$

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  be Laurent polynomial algebras with respective variables.
- For $0 \leq i \leq n$, let ξ_i denote a m_i-th primitive root of unity. Let

$$\mathfrak{g}(\overline{k}_0, \overline{k}) = \{X \in \mathfrak{g} \,|\, \sigma_i X = \xi_i^{k_i} X, 0 \leq i \leq n\}.$$

  Then $\bigoplus_{(k_0,k) \in \mathbb{Z}^{n+1}} \mathfrak{g}(\overline{k}_0, \overline{k}) t_0^{k_0} t^k$ is called multiloop algebra.

Lie torus

Let $\mathfrak{g}_1$ be any Lie algebra and $\mathfrak{h}_1$ be its finite dimensional ad-diagonalizable subalgebra. We set for $\alpha \in \mathfrak{h}_1^*$,

$$\mathfrak{g}_{1,\alpha} = \{x \in \mathfrak{g}_1 \mid [h, x] = \alpha(h)x, h \in \mathfrak{h}_1^*\}.$$

Then we have

$$\mathfrak{g}_1 = \bigoplus_{\alpha \in \mathfrak{h}_1^*} \mathfrak{g}_{1,\alpha}.$$

Let $\Delta(\mathfrak{g}_1, \mathfrak{h}_1) = \{\alpha \in \mathfrak{h}_1^* \mid \mathfrak{g}_{1,\alpha} \neq 0\}$ which includes 0. Let $\Delta^\times(\mathfrak{g}_1, \mathfrak{h}_1) = \Delta(\mathfrak{g}_1, \mathfrak{h}_1) \backslash 0$.
For a finite dimensional simple Lie algebra $\mathfrak{g}_1$, $\Delta_1^\times = \Delta(\mathfrak{g}_1, \mathfrak{h}_1)^\times$ is an irreducible reduced finite root system with atmost two root lenths. Let $\Delta_{1,\mathrm{sh}}^\times$ denote the set of non-zero short roots.

$$\Delta_{1,\mathrm{en}}^\times = \begin{cases} \Delta_1^\times \cup 2\Delta_{1,\mathrm{sh}}^\times, & \text{if } \Delta_1^\times \text{of} \quad \text{type} B_l \\ \Delta_1^\times, & \text{otherwise} \end{cases}$$

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1. $\mathfrak{g}(\overline{0}, \overline{0})$ *is a simple Lie algebra.*
2. *As* $\mathfrak{g}(\overline{0}, \overline{0})$ *module, each* $\mathfrak{g}(\overline{k}_0, \overline{k}) = U(\overline{k}_0, \overline{k}) \oplus V(\overline{k}_0, \overline{k})$ *where* $U(\overline{k}_0, \overline{k})$ *is trivial module and either* $V(\overline{k}_0, \overline{k})$ *is zero or satisfy the property* (M)*.*

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3. $|\langle \sigma_0, \ldots, \sigma_n \rangle| = \prod_{i=0}^{n} |\sigma_i|$.

Universal central extension of Lie torus

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- Let $LT = \bigoplus_{(k_0,k)\in\mathbb{Z}^{n+1}} \mathfrak{g}(\overline{k}_0, \overline{k}) t_0^{k_0} t^k$ be a Lie torus.
- Let Ω_A be the vector space spanned by symbols $t_0^{k_0} t^k K_i, 0 \leq i \leq n, k_0 \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$. Let dA be the subspace of Ω_A spanned by $\sum_{i=0}^{n} k_i t_0^{k_0} t^k K_i$. Let $Z = \Omega_A / dA$.

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- Similarly define $Z(n) = \Omega_{A_n}/dA_n,\ Z(m) = \Omega_{A(m)}/dA(m)$ and $Z(m_0, m) = \Omega_{A(m_0,m)}/dA(m_0, m)$.

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- To see above bracket is closed, notice that $(X|Y) \neq 0 \Rightarrow k + l \in \Gamma$ and $k_0 + l_0 \in \Gamma_0$. This follows from the standard fact that $(\cdot|\cdot)$ is invariant under $\sigma_i, 0 \leq i \leq n$.

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Proposition (J.Sun - 2009)

$\overline{LT}$ is the universal central extension of LT.

- Both LT and $\overline{LT}$ are naturally $\mathbb{Z}^{n+1}$ graded. To reflect this fact we add derivations. Let $\overline{D}$ be the space spanned by $d_0, d_1, \cdots, d_n$.

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- Let $\widetilde{LT} = \overline{LT} \oplus \overline{D}$. Extend the Lie bracket in the following way:

$$\begin{aligned} [d_i, X(k_0, k)] &= k_i X(k_0, k); \\ [d_i, t_0^{k_0} t^k K_j] &= k_i t_0^{k_0} t^k K_j; \\ [d_i, d_j] &= 0. \end{aligned}$$

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\end{aligned}$$

- The purpose of this work is to classify irreducible integrable weight modules for $\widetilde{LT}$.

- Let $\mathfrak{h}(0)$ be a Cartan subalgebra of $\mathfrak{g}(\bar{0},\bar{0})$
- Let $\widetilde{\mathfrak{h}} = \mathfrak{h}(0) \oplus \sum_{0 \le i \le n} \mathbb{C}K_i \oplus \overline{D}$ which is an abelian subalgebra of $\widetilde{LT}$.
- $\delta_i \in \widetilde{\mathfrak{h}}^*$ such that $\delta_i(\mathfrak{h}(0)) = \delta_i(K_j) = 0$ and $\delta_i(d_j) = \delta_{ij}, 0 \le i,j \le n$. For $(k_0, k) \in \mathbb{Z}^{n+1}$, let $\delta_k = \sum k_i \delta_i$ and $\delta(k_0,k) = k_0\delta_0 + \delta_k$.
-

$$\widetilde{LT}_{\alpha+\delta(k_0,k)} = \begin{cases} \mathfrak{g}(\overline{k}_0, \overline{k}, \alpha) \otimes t_0^{k_0} t^k, \alpha \neq 0, \\ \mathfrak{g}(\overline{k}_0, \overline{k}, 0) \otimes t_0^{k_0} t^k \oplus \sum\limits_{\substack{\overline{k}_0, \overline{k} \in \Gamma_0 \oplus \Gamma \\ 0 \le i \le n}} t^{k_0} t^k K_i, \ \alpha = 0 \text{ and } (k_0, k \\ \\ \widetilde{\mathfrak{h}} = \mathfrak{g}(\overline{0}, \overline{0}, 0) \oplus \sum\limits_{0 \le i \le n} \mathbb{C}K_i \oplus \overline{D}, \ (k_0, k, \alpha) = (0,0,0). \end{cases}$$

- Then $\widetilde{LT} = \bigoplus_{\substack{\alpha \in \mathfrak{h}(0)^* \\ (k_0,k) \in \mathbb{Z}^{n+1}}} \widetilde{LT}_{\alpha+\delta(k_0,k)}$ is a root space decomposition with respect to $\widetilde{\mathfrak{h}}$ and each root space is finite dimensional.
- A module V of $\widetilde{LT}$ is called weight module if
  (a) $V = \bigoplus_{\lambda \in \widetilde{\mathfrak{h}}^*} V_\lambda, V_\lambda = \{v \in V \mid hv = \lambda(h)v, h \in \widetilde{\mathfrak{h}}\}$.
  (b) $\dim V_\lambda < \infty$.
- A root $\alpha + \delta(k_0, k)$ of $\widetilde{LT}$ is called real root if $\alpha \neq 0$ and null root if $\alpha = 0$.
- A weight module V of $\widetilde{LT}$ is called integrable if every real root vector acts locally nilpotently on V, i.e., for $X \in \widetilde{LT}_{\alpha+\delta(k_0,k)}, \alpha \neq 0$ and $v \in V$, there exists $b = b(v, \alpha + \delta(k_0, k))$ such that $X^b.v = 0$.

Subquotient of $\widetilde{LT}$

Consider a Lie algebra $\mathfrak{L} = LT \oplus \mathbb{C}K \otimes A(m) \oplus \mathbb{C}d_0$ where K is a symbol. Let $X(k_0, k) \in \mathfrak{g}(\overline{k}_0, \overline{k})$ and $Y(l_0, l) \in \mathfrak{g}(\bar{l}_0, \bar{l})$ and define the bracket operations on $\mathfrak{L}$ as follows:

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- Let $\widetilde{\mathfrak{L}} = \mathfrak{L} \oplus D$ where D is the space spanned by derivations $d_1, d_2, \cdots, d_n$. Extend the Lie bracket to $\widetilde{\mathfrak{L}}$ by defining D action on $\mathfrak{L}$ as before.

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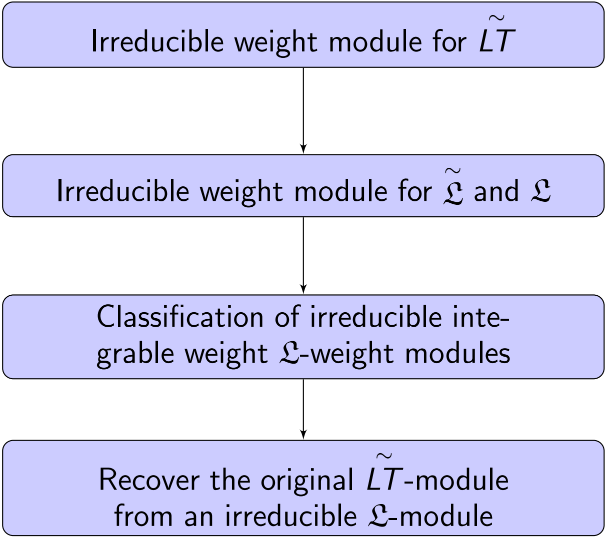
- Then $\Phi : \widetilde{LT} \to \widetilde{\mathfrak{L}}$ by

$$\begin{aligned}
\Phi X(k_0, k) &= X(k_0, k), X \in \mathfrak{g}(k_0, k);\\
\Phi(t_0^{k_0} t^k K_i) &= 0 \text{ if } i \neq 0 \text{ or } k_0 \neq 0;\\
\Phi(t^k K_0) &= K \otimes t^k;\\
\Phi(d_i) &= d_i, 0 \leq i \leq n.
\end{aligned}$$

is a Lie algebra homomorphism.


Irreducible weight module for $\widetilde{LT}$
Irreducible weight module for $\widetilde{\mathfrak{L}}$ and $\mathfrak{L}$
Classification of irreducible integrable weight $\mathfrak{L}$-weight modules
Recover the original $\widetilde{LT}$-module from an irreducible $\mathfrak{L}$-module

Highest weight irreducible modules for $\mathfrak{L}$

- Let $\widetilde{\mathfrak{h}}(0) = \mathfrak{h}(0) \oplus \mathbb{C}K \oplus \mathbb{C}d_0$, and $\mathfrak{L} = \bigoplus_{\substack{\alpha \in \mathfrak{h}(0)^* \\ k_0 \in \mathbb{Z}}} \mathfrak{L}_{\alpha + k_0\delta}$ is a root space decomposition with respect to $\widetilde{\mathfrak{h}}(0)$, where

$$\mathfrak{L}_{\alpha+k_0\delta} = \begin{cases} \bigoplus_{k \in \mathbb{Z}^n} \mathfrak{g}(k_0, k, \alpha) t_0^{k_0} t^k, & \text{if } \alpha + k_0\delta \neq 0, \\ \bigoplus_{k \in \mathbb{Z}^n} \mathfrak{g}(0, \overline{k}, 0) t^k \oplus K \otimes A(m) \oplus \mathbb{C}d_0, & \text{if } \alpha + k_0\delta = 0. \end{cases}$$

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- Here $\delta \in \widetilde{\mathfrak{h}}(0)^*$ such that $\delta(\mathfrak{h}(0)) = 0, \delta(K) = 0$ and $\delta(d_0) = 1$.

$$\widetilde{\Delta}_{+} = \{\alpha + k_0\delta \in \widetilde{\Delta} \mid k_0 > 0 \text{ or } k_0 = 0, \alpha > 0\}$$
$$\widetilde{\Delta}_{-} = \{\alpha + k_0\delta \in \widetilde{\Delta} \mid k_0 < 0 \text{ or } k_0 = 0, \alpha < 0\}.$$

$$\mathfrak{L}^{+} = \bigoplus_{\alpha + k_0\delta > 0} \mathfrak{L}_{\alpha + k_0\delta},$$
$$\mathfrak{L}^{-} = \bigoplus_{\alpha + k_0\delta < 0} \mathfrak{L}_{\alpha + k_0\delta}, \text{ and}$$
$$\mathfrak{L}^{0} = \mathfrak{L}_{0}.$$

Construction of highest weight module for $\mathfrak{L}$

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- Let $U(\mathfrak{L})$ denote the universal enveloping algebra of $\mathfrak{L}$. Define Verma module for $\mathfrak{L}$.

$$M(N) = U(\mathfrak{L}) \bigotimes_{\mathfrak{L}^+ \oplus \mathfrak{L}^0} N$$

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- By standard arguments, $M(N)$ admits a unique irreducible quotient say $V(N)$.
- $M(N)$ and $V(N)$ are weight module with respect to $\widetilde{\mathfrak{h}}(0)$. But they may not have finite dimensional weight spaces.

Necessary and sufficient condition for $V(N)$ to have finite dimensional weight spaces

- Let I be an ideal in $A(m)$ and let

$$\mathfrak{L}^0(I) = \bigoplus_{k \in \mathbb{Z}^n} \mathfrak{g}(0, \overline{k}, 0) t^k I \quad \oplus \quad K \otimes I,$$

$$\mathfrak{L}(I) = \bigoplus_{\substack{k_0 \in \mathbb{Z} \\ k \in \mathbb{Z}^n}} \mathfrak{g}(\overline{k}_0, \overline{k}) \otimes t_0^{k_0} t^k I \quad \oplus \quad K \otimes I.$$

- Then $\mathfrak{L}^0(I)$ is an ideal in $\mathfrak{L}^0$ and $\mathfrak{L}(I)$ is an ideal in $\mathfrak{L}$.

Necessary and sufficient condition for $V(N)$ to have finite dimensional weight spaces

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$$\mathfrak{L}^0(I) = \bigoplus_{k \in \mathbb{Z}^n} \mathfrak{g}(0, \overline{k}, 0) t^k I \quad \oplus \quad K \otimes I,$$

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- Then $\mathfrak{L}^0(I)$ is an ideal in $\mathfrak{L}^0$ and $\mathfrak{L}(I)$ is an ideal in $\mathfrak{L}$.

Proposition

Suppose N is finite dimensional irreducible module for $\mathfrak{L}^0$ such that $\mathfrak{L}^0(I).N = 0$ for some ideal I of $A(m)$. Then $\mathfrak{L}(I).V(N) = 0$.

Theorem (1)

$V(N)$ has finite dimensional weight spaces with respect to $\widetilde{\mathfrak{h}}(0)$ if and only if there exists a co-finite ideal I of $A(m)$ such that $\mathfrak{L}^0(I).N = 0$.

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- Let write $P_i(t_i^{m_i}) = \prod_{j=1}^{q_i} (t_i^{m_i} - a_{ij}^{m_i})^{b_{ij}}$ for some positive integers b_{ij} and q_i, with $a_{ij}^{m_i} \neq a_{ij'}^{m_i}$ for $j \neq j'$.

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- Let $P_i'(t_i^{m_i}) = \prod_{j=1}^{q_i} (t_i^{m_i} - a_{ij}^{m_i})$ and $I' = \langle P_i' \rangle$
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Theorem (2)

Let $V(N)$ be an integrable module for $\mathfrak{L}$. Then $\mathfrak{L}(I')V(N) = 0$.

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- Define

$$\mathfrak{L}(\mathfrak{g}, \sigma_0) = \bigoplus_{k_0 \in \mathbb{Z}} \mathfrak{g}_{\overline{k_0}} \otimes t^{k_0} \oplus \mathbb{C}K$$

which is known to be an affine Lie algebra.

Proposition

$\mathfrak{L}'/\mathfrak{L}'(I') \cong \bigoplus_{N-copies} \mathfrak{L}(\mathfrak{g}, \sigma_0)$, where $\mathfrak{L}' = \bigoplus_{(k_0,k) \in \mathbb{Z}^n} \mathfrak{g}(\overline{k}_0, \overline{k}) t_0^{k_0} t^k \oplus K \otimes A(m)$ and $\mathfrak{L}'(I')$ defined similarly as before.

- Let $\mathfrak{g}_{aff} = \mathfrak{g}(\overline{0}, \overline{0}) \otimes \mathbb{C}[t_0^{m_0}, t_0^{-m_0}] \oplus \mathbb{C}K \oplus \mathbb{C}d_0$ which is a subalgebra of $\mathfrak{L}$.

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Theorem

Suppose V is an irreducible module for $\mathfrak{L}$ with finite dimensional weight spaces with respect to $\widetilde{\mathfrak{h}}(0)$. Further, suppose V is integrable for $\mathfrak{g}_{aff}$ where the canonical central element m_0K acts as positive integer, then V is an highest weight module for $\mathfrak{L}$.

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- Now as V is an highest weight module, let λ be the top weight and put $N = V_\lambda$. Since V is irreducible, by weight argument we see that N is irreducible $\mathfrak{L}^0$- module. Thus $V \cong V(N)$.

Theorem (S.E. Rao, -)

Let V be an irreducible integrable $\widetilde{LT}$-module with center acting non-trivially. Then V is a highest weight module for finitely many copies of affine Lie algebra upto an automorphism.

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