Based Quantum Cluster Algebras

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Interactions of quantum affine algebras with cluster algebras, current algebras, and categorification

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Motivation

Fix $d \in \mathbb{Z}_{>0}$, let $q^{\frac{1}{2d}}$ be an indeterminate, and set $\mathbb{K} = \mathbb{Q}(q^{\frac{1}{2d}})$.

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Definition

For any integers $m, n \geq 1$, $\mathcal{A}_q[Mat_{m,n}]$ is the $\mathbb{K}$-algebra generated by symbols $\{x_{i,j} \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ and subject to the relations

$$\begin{aligned}
x_{i,\ell}x_{i,j} &= qx_{i,j}x_{i,\ell} && (j < \ell),\\
x_{k,j}x_{i,j} &= qx_{i,j}x_{k,j} && (i < k),\\
x_{k,j}x_{i,\ell} &= x_{i,\ell}x_{k,j} && (i < k,\ j < \ell),\\
x_{k,\ell}x_{i,j} &= x_{i,j}x_{k,\ell} + (q - q^{-1})x_{i,\ell}x_{k,j} && (i < k,\ j < \ell).
\end{aligned}$$

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Theorem (Kang-Kashiwara-Kim-Oh '15)

If $\mathfrak{g}$ is of type ADE, the quantum cluster monomials of every quantum Schubert cell are elements of the dual canonical basis.

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Given any two $U_q(\mathfrak{g})$-module algebras A and B satisfying certain conditions, there is a natural way to make their tensor product (as vector spaces) into a $U_q(\mathfrak{g})$-module algebra: the braided tensor product $A \underline{\otimes} B$.

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Proposition

$$\mathcal{A}_q[Mat_{m,n_1}] \underline{\otimes} \mathcal{A}_q[Mat_{m,n_2}] \cong \mathcal{A}_q[Mat_{m,n_1+n_2}].$$

"Definition"

A quantum cluster algebra is a $\mathbb{K}$-algebra with a distinguished set of generators, called (quantum) cluster variables. Cluster variables are gathered into maximal (overlapping) subsets called clusters, the elements of which pairwise commute up to powers of q. Every cluster is reachable from any other cluster by a series of single-variable exchanges, called mutations.

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- Every quantum cluster algebra has a unique anti-linear algebra anti-involution which fixes the cluster variables (usually denoted by $z \mapsto \overline{z}$).
- Quantum cluster monomials are monomials in the cluster variables of a single cluster, scaled by the power of $q^{\frac{1}{2d}}$ so that they are fixed under the bar.

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- $U_q(\mathfrak{g})$ has a "universal $\mathcal{R}$-matrix", which braids the category of weight modules on which E_i acts locally nilpotently for each $i \in I$. $\mathcal{R}$ satisfies "$\overline{\mathcal{R}} = \mathcal{R}^{-1}$".

Definition

A *barred module algebra* is a pair $(A, \bar{\ })$, where A is a $U_q(\mathfrak{g})$-module algebra and $\bar{\ } : A \to A$ is an anti-linear algebra anti-involution such that $\overline{u(a)} = \overline{u}(\overline{a})$ for $u \in U_q(\mathfrak{g})$ and $a \in A$.

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Example

$(\mathcal{A}_q[Mat_{m,n}], \bar{\ })$ is a barred module over $U_q(\mathfrak{sl}_m)$, where $\bar{\ } : \mathcal{A}_q[Mat_{m,n}] \to \mathcal{A}_q[Mat_{m,n}]$ is the unique anti-linear algebra anti-involution such that $\overline{x_{i,j}} = x_{i,j}$.

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Furthermore, the dual canonical basis is fixed under the bar.

Theorem (S)

Given barred module algebras $(A, \bar{\ })$ *and* $(A', \bar{\ })$, *there is a unique barred module algebra structure* $(A \underline{\otimes} A', \bar{\ })$ *so that* $\overline{a \otimes 1} = \overline{a} \otimes 1$ *and* $\overline{1 \otimes a'} = 1 \otimes \overline{a'}$ *for* $a \in A$ *and* $a' \in A'$.

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Caution: It is almost never the case that $\overline{a \otimes a'} = \overline{a} \otimes \overline{a'}$. Namely,

$$\overline{a \otimes a'} = \overline{(a \otimes 1)(1 \otimes a')} = (1 \otimes \overline{a'})(\overline{a} \otimes 1).$$

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If $\mathcal{B}$ and $\mathcal{B}'$ are bar-invariant bases of barred module algebras $(A, \bar{\ })$ and $(A', \bar{\ })$ satisfying certain criteria, there is a canonical choice $\mathcal{B} \diamond \mathcal{B}'$ of bar-invariant basis of $(A \underline{\otimes} A', \bar{\ })$ satisfying the same criteria.

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If $\mathcal{B}$ and $\mathcal{B}'$ are bar-invariant bases of barred module algebras $(A, ^{-})$ and $(A', ^{-})$ satisfying certain criteria, there is a canonical choice $\mathcal{B} \diamond \mathcal{B}'$ of bar-invariant basis of $(A \underline{\otimes} A', ^{-})$ satisfying the same criteria.

Let $\mathcal{B}_{m,n}$ be the dual canonical basis for $\mathcal{A}_q[Mat_{m,n}]$.

Theorem (S)

The $U_q(\mathfrak{g})$-module algebra isomorphism

$$\mathcal{A}_q[Mat_{m,n_1}] \underline{\otimes} \mathcal{A}_q[Mat_{m,n_2}] \cong \mathcal{A}_q[Mat_{m,n_1+n_2}]$$

carries $\mathcal{B}_{m,n_1} \diamond \mathcal{B}_{m,n_2}$ to $\mathcal{B}_{m,n_1+n_2}$.

Example

$\mathcal{A}_q[U]$, $\mathcal{A}_q[B]$, and $\mathcal{A}_q[G/U]$ are certain $U_q(\mathfrak{g})$-module algebras, generated (as module algebras) by $\{x_i \mid i \in I\}$, $\{v_i^{\pm 1} \mid i \in I\}$, and $\{v_i \mid i \in I\}$, respectively.

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Each has a unique anti-linear algebra anti-involution fixing these generators and giving it a barred module algebra structure. In fact, these are all quantum cluster algebras with the anti-linear anti-involutions fixing the quantum cluster monomials.

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Theorem (Berenstein-S)

$\mathcal{A}_q[U]^{\underline{\otimes} n} \cong \mathcal{A}_q[U]^{\overline{\otimes} n}$ and $\mathcal{A}_q[B]^{\underline{\otimes} n} \cong \mathcal{A}_q[B]^{\overline{\otimes} n}$ (as algebras).

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Corollary

$\mathcal{A}_q[U]^{\underline{\otimes} n}$ and $\mathcal{A}_q[B]^{\underline{\otimes} n}$ are quantum cluster algebras.

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Example (S)

$\mathcal{A}_q[SL_3/U]$ is a quantum cluster algebra of type A_1, so the cluster monomials form a basis (call it $\mathcal{B}$).

$\mathcal{A}_q[SL_3/U]\underline{\otimes}\mathcal{A}_q[SL_3/U]$ is a quantum cluster algebra of type D_4 and $\mathcal{B} \diamond \mathcal{B}$ coincides with the set of cluster monomials.

Thank You!