Weyl orbits of π-systems in Kac-Moody algebras

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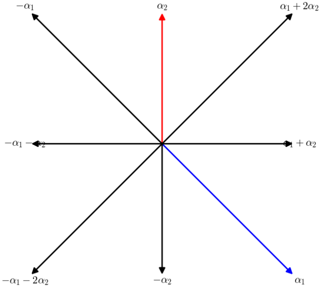
June 4, 2018

π-systems

Definition: Let Δ be a root system. A non-empty subset $\Sigma \subset \Delta$ is called a π-system if $\alpha - \beta \notin \Delta$ for all $\alpha, \beta \in \Sigma$.



π-systems of B_2:

- Singletons
- Two roots making a $3\pi/4$ angle (simple roots of Δ)
- Two long roots making a $\pi/2$ angle.
- Two roots making a π angle
- Two long and one short root, making angles $3\pi/4, 3\pi/4, \pi/2$

π-systems give rise to GCMs

Definition: An integer matrix C is a *generalized Cartan matrix* (GCM) if (i) $c_{ii} = 2$ $(1 \leq i \leq m)$ (ii) $c_{ij} \leq 0$ for $i \neq j$ (iii) $c_{ij} = 0$ iff $c_{ji} = 0$.

Lemma: Let $\Sigma = \{\beta_1, \beta_2, \cdots, \beta_m\}$ be a π-system in Δ. Then the matrix $B = \left[\frac{2(\beta_i|\beta_j)}{(\beta_i|\beta_i)}\right]_{ij}$ is a GCM. □

Examples for $\Delta = B_2$:

- Singletons: $B = [2]$;
- Simple roots of Δ: $B = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix} =: A(\Delta)$.
- Two long roots making a $\pi/2$ angle: $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.
- Two roots making a π angle: $B = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$
- Two long and one short root, making angles $3\pi/4, 3\pi/4, \pi/2$:

$$B = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

π-systems in Kac-Moody algebras

Let A be a GCM, and $\mathfrak{g}(A)$ the corresponding Kac-Moody algebra.

$$\mathfrak{g}(A) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(A)} \mathfrak{g}_\alpha$$

- $\Delta(A) = \Delta^{re}(A) \sqcup \Delta^{im}(A)$
- $\mathfrak{g}(A)_\alpha$ are one-dimensional for real roots α.
- $\Sigma \subset \Delta^{re}(A)$ is a π-system if $\alpha - \beta \notin \Delta(A)$ for all $\alpha, \beta \in \Sigma$.

Theorem

- *Let A be a GCM and $\{\beta_i\}_{i=1}^m \subset \Delta^{re}(A)$ be a π-system in $\Delta(A)$ with GCM B.*
- *Let $e_{\pm\beta_i} \in \mathfrak{g}'(A)_{\pm\beta_i}$ such that $[e_{\beta_i}, e_{-\beta_i}] = \beta_i^\vee$.*

Then:

- *there exists a unique Lie algebra homomorphism $i : \mathfrak{g}'(B) \to \mathfrak{g}'(A)$ such that $e_i \mapsto e_{\beta_i}, f_i \mapsto e_{-\beta_i}, \alpha_i^\vee \mapsto \beta_i^\vee$.*
- *this map is injective iff the π-system is linearly independent.*

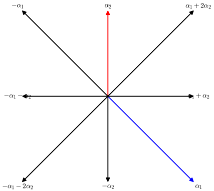
π-systems in Kac-Moody algebras

Lemma

Conversely, given a Lie algebra homomorphism $\phi : \mathfrak{g}'(B) \to \mathfrak{g}'(A)$ satisfying $0 \neq \phi(e_i) \in \mathfrak{g}'(A)_{\beta_i}$, $0 \neq \phi(f_i) \in \mathfrak{g}'(A)_{-\beta_i}$ for some real roots β_i of $\mathfrak{g}'(A)$. Then, the set $\{\beta_i\}$ is a π-system of type B in A.

- The image of $\mathfrak{g}'(B)$ will be called a *regular subalgebra* of $\mathfrak{g}'(A)$.
- If A has a π-system of type B and B has a π-system of type C, then A has a π-system of type C.

Weyl conjugacy of π-systems



- Consider π-systems in Δ with a fixed GCM, say $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.
- Pair of orthogonal long roots.
- Any two such π-systems can be transformed one into the other by a *reflection associated to one of the roots* in Δ.
- Let $W(\Delta)$ be the group generated by reflections associated to the roots (the Weyl group of Δ).
- In this case, $W(\Delta)$ acts transitively on such π-systems.

General Question (in the setting of Kac-Moody algebras)

Let A be a GCM. Consider the set of all π-systems in A of a fixed type B. How many $W(A)$-orbits does this split into ?

Theorem

Let A, B be symmetrizable GCMs and Σ a linearly independent π-system of type B in A. If B is indecomposable*, then:*

1. *There exists $w \in W(A)$ such that $w\Sigma \subset \Delta_+^{re}(A)$ or $w\Sigma \subset \Delta_-^{re}(A)$.*
2. *There exist $w_1, w_2 \in W(A)$ such that $w_1\Sigma \subset \Delta_+^{re}(A)$ and $w_2\Sigma \subset \Delta_-^{re}(A)$ if and only if B is of finite type.*

- Let $\mathfrak{m}(B, A)$ denote the number of $W(A)$-orbits of π-systems in A of type B (this could be infinity in general).
- When A, B are of finite type, Dynkin determined all these numbers $\mathfrak{m}(B, A)$.

Dynkin: On Semisimple subalgebras of semisimple Lie algebras (1951)

Table 9.
Regular semisimple subalgebras of the classical algebras

Algebra	Types of subalgebras	
A_n	$A_{k_1} + \ldots + A_{k_s}$	$\left(\sum_{i=1}^{s} (k_i + 1) = n + 1 \right)$
B_n	$A_{k_1} + \ldots + A_{k_s} + D_{m_1} + \ldots + D_{m_r} + B_m$	$\left(\sum_{i=1}^{s} (k_i + 1) + \sum_{i=1}^{r} m_i + m = n \right)$
C_n	$A_{k_1} + \ldots + A_{k_s} + C_{l_1} + \ldots + C_{l_r}$	$\left(\sum_{i=1}^{s} (k_i + 1) + \sum_{i=1}^{r} l_i = n \right)$
D_n	$A_{k_1} + \ldots + A_{k_s} + D_{m_1} + \ldots + D_{m_r}$	$\left(\sum_{i=1}^{s} (k_i + 1) + \sum_{i=1}^{r} m_i = n \right)$

$k_1 \geqslant k_2 \geqslant \ldots \geqslant k_s \geqslant 0, \quad l_1 \geqslant l_2 \geqslant \ldots \geqslant l_r > 0, \quad m_1 \geqslant m_2 \geqslant \ldots \geqslant m_r > 1, \quad m \geqslant 0$

For the classical algebras A_n, B_n, C_n, D_n, the types of regular semisimple subalgebras are given in table 9. Every subalgebra is conjugate to one of these types. An exception occurs in the case of the types of subalgebras of D_n which correspond to odd $k_1, k_2, \cdots, k_s$ and $r = 0$. To every such type there correspond two classes of conjugate subalgebras, transformable one into the other by an outer automorphism of D_n.

Dynkin (1951): the exceptional Lie algebras

A_2	A_1
$A_1 + \widetilde{A}_1$	$\widetilde{A}_1$

F_4

B_4	$2A_1 + \widetilde{A}_1$
$A_3 + \widetilde{A}_1$	$A_1 + \widetilde{A}_2$
$A_2 + \widetilde{A}_2$	C_3
$C_3 + A_1$	$3A_1$
D_4	A_2
$B_2 + 2A_1$	B_2
$4A_1$	$A_1 + \widetilde{A}_1$
B_3	$2A_1$
$B_2 + A_1$	$\widetilde{A}_2$
$A_2 + \widetilde{A}_1$	$\widetilde{A}_1$
A_3	A_1

$A_5 + A_1$	$A_2 + 2A_1$
$3A_2$	$4A_1$
A_5	D_4
$2A_2 + A_1$	A_3
$A_4 + A_1$	$A_2 + A_1$
D_5	$3A_1$
$A_3 + 2A_1$	A_2
A_4	$2A_1$
$A_3 + A_1$	A_1
$2A_2$	

$D_6 + A_1$	$3A_2$	D_4
$A_5 + A_2$	$2A_3$	A_4
$2A_3 + A_1$	A_6	$[A_3 + A_1]'$
A_7	$6A_1$	$[A_3 + A_1]''$
$D_4 + 3A_1$	D_5	$2A_2$
$7A_1$	$A_4 + A_1$	$A_2 + 2A_1$
E_6	$2A_2 + A_1$	$[4A_1]'$
$D_5 + A_1$	$[A_5]'$	$[4A_1]''$
$A_4 + A_2$	$[A_5]''$	A_3
$A_3 + A_2 + A_1$	$D_4 + A_1$	$A_2 + A_1$
$[A_5 + A_1]'$	$A_3 + A_2$	$[3A_1]'$
$[A_5 + A_1]''$	$5A_1$	$[3A_1]''$
D_6	$A_2 + 3A_1$	A_2
$D_4 + 2A_1$	$[A_3 + 2A_1]'$	$2A_1$
$A_3 + 3A_1$	$[A_3 + 2A_1]''$	A_1

A_8	$A_4 + A_2 + A_1$	$A_4 + A_3$	$A_4 + 2A_1$	$A_2 + 3A_1$
D_8	$A_5 + A_2$	$A_5 + 2A_1$	A_6	$2A_2 + A_1$
$A_7 + A_1$	$3A_2 + A_1$	$[A_7]'$	$A_3 + A_2 + A_1$	D_4
$A_5 + A_2 + A_1$	$E_6 + A_1$	$[A_7]''$	$[A_5 + A_1]'$	$[4A_1]'$
$2A_4$	E_7	$3A_2$	$[A_5 + A_1]''$	$[4A_1]''$
$4A_2$	D_7	E_6	$A_4 + A_2$	$A_2 + 2A_1$
$A_6 + A_2$	$D_5 + 2A_1$	D_6	$2A_2 + 2A_1$	$2A_2$
$A_7 + A_1$	$D_4 + 3A_1$	$D_4 + 2A_1$	D_5	$A_3 + A_1$
$D_6 + 2A_1$	$2A_3 + A_1$	$[2A_3]'$	$[A_3 + 2A_1]'$	A_4
$D_5 + A_3$	$7A_1$	$[2A_3]''$	$[A_3 + 2A_1]''$	A_3
$2D_4$	$D_6 + A_1$	$D_5 + A_1$	$A_3 + A_2$	$A_2 + A_1$
$D_4 + 4A_1$	$D_5 + A_2$	$A_3 + 3A_1$	A_5	$3A_1$
$2A_3 + 2A_1$	$A_3 + A_2 + 2A_1$	$D_4 + A_2$	$5A_1$	A_2
$8A_1$	$D_4 + A_3$	$6A_1$	$A_4 + A_1$	$2A_1$
$A_6 + A_1$	$A_3 + 4A_1$	$A_2 + 4A_1$	$D_4 + A_1$	A_1

π-systems of affine type

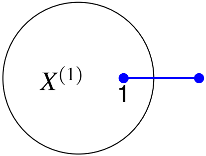
Theorem

Let A be a symmetrizable GCM and B be a GCM of affine type. Suppose Σ is a linearly independent π-system of type B in A. Then,

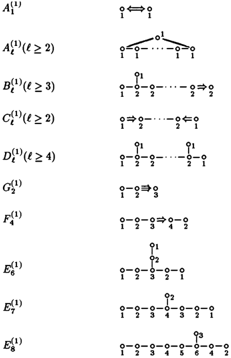
1. *There exists an affine subdiagram Y of $S(A)$ and $w \in W(A)$ such that every element of $w\Sigma$ is supported in Y.*
2. *Suppose (Y', w') is another such pair, i.e., with Y' a subdiagram of affine type, $w' \in W(A)$ such that $w'\Sigma$ is supported in Y'. Then $Y = Y'$ and $w'w^{-1} \in W(Y \sqcup Y^{\perp})$.*
3. *$\mathfrak{m}(B, A) = \infty$.*

Let A be a symmetrizable GCM such that $S(A)$ has no subdiagrams of affine type. Then A contains no linearly independent π-systems of affine type.

Overextended Dynkin diagrams (*EXT* type)



where $X^{(1)}$ is untwisted affine (i.e, one of the following) and the marked vertex is *special*. We denote this diagram X^{++}.



The main theorem in general

Theorem

Let X be a simply-laced Dynkin diagram ($\leftrightarrow$ symmetric GCM) and let K be a simply-laced Dynkin diagram of EXT type. Then:

1. *There exists a π-system in X of type K if and only if there exists an EXT type subdiagram Z of X such that Z° has a π-system of type K°.*
2. *The number of $W(X)$ orbits of π-systems of type K in X is given by:*

$$\mathfrak{m}\,(K,X) = 2 \sum_{\substack{Z\subseteq X \\ Z\in\overline{\mathrm{EXT}}}} \mathfrak{m}(K^\circ, Z^\circ) \tag{1}$$

where K°, Z° denote their finite parts.

If $K = X^{++}$, then we write $X = K^\circ$.

Corollaries

- The theorem reduces the computation of the multiplicity of K in X to a sum of multiplicities involving only finite type diagrams.
- The latter are completely known (Dynkin).

Corollary

Let K be a simply-laced Dynkin diagram of EXT type. Then,

1. $\mathfrak{m}(K,X)$ *is finite for all simply-laced diagrams* X.
2. $\mathfrak{m}(K,X) = 2\,\mathfrak{m}(K^\circ, X^\circ)$ *for all* $X \in \mathrm{HYP} \cap \mathrm{EXT}$.

The main theorem: a special case

$A_1{}^{++}$:

GCM: $B = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

For $K = A_1^{++}$, K° is of type A_1. Since any Z° occurring on the right hand side of (1) is simply laced, we have $\mathfrak{m}(K^\circ, Z^\circ) = 1$. So this reduces exactly to the following theorem:

Theorem

Let X be a simply laced Dynkin diagram ($\leftrightarrow$ symmetric GCM). Then:

1. *X has a π-system of type A_1^{++} if and only if it contains a subdiagram of EXT type.*
2. *The number of $W(X)$-orbits of π-systems of type A_1^{++} in A is twice the number of such subdiagrams (and is, in particular, finite).*

In particular, if X is itself (simply laced) of EXT type, then X contains a π-system of type A_1^{++}.

Thank You