Trace and center of the twisted Heisenberg category

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Trace decategorification

The trace (or zeroth Hochschild homology) of a $\mathbb{C}$-linear additive category $\mathcal{C}$:

$$\mathrm{Tr}(\mathcal{C}) := \left(\oplus_{x \in \mathrm{ob}(\mathcal{C})} \mathrm{End}_{\mathcal{C}}(X) \right) \Big/ \mathrm{Span}\{fg - gf\},$$

where f and g run through all pairs of morphisms $f : x \to y$ and $g : y \to x$ with $x, y \in \mathrm{Ob}(\mathcal{C})$.

If $\mathcal{C}$ is monoidal, then $\mathrm{Tr}(\mathcal{C})$ is an algebra.

Prototype: Heisenberg category

[Cautis-Lauda-Licata-Sussan 2015] show

$$\mathrm{Tr}(\mathcal{H}) \cong W_{1+\infty}/\langle C - 1, w_{0,0}\rangle.$$

[Kvinge, Licata, Mitchell 2017] show the center, $\mathrm{End}_{\mathcal{H}}(\mathbb{1})$, of the Heisenberg category is isomorphic to the shifted symmetric functions Λ^*.

Objective

Twisted Heisenberg algebra $\mathfrak{h}_{tw}$: associative algebra with generators $h_{m/2}$, $m \in 2\mathbb{Z}+1$ subject to

$$[h_{\frac{n}{2}}, h_{\frac{m}{2}}] = \frac{n}{2}\delta_{n,-m}.$$

Categorified by twisted Heisenberg category $\mathcal{H}_{tw}$[Cautis-Sussan, 2015]:

$$\mathfrak{h}_{tw} \subset K_0(\mathcal{H}_{tw}).$$

Conjecturally isomorphic.

We will describe Tr (and center) of $\mathcal{H}_{tw}$.

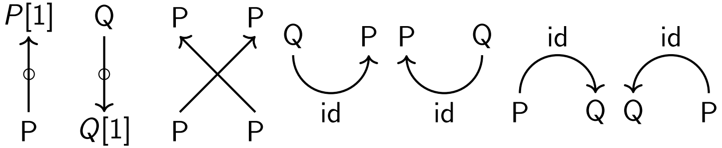
Twisted Heisenberg category

Twisted Heisenberg category $\mathcal{H}_{tw}$:
generating objects:

$$\mathrm{P} = \uparrow \qquad \mathrm{Q} = \downarrow$$

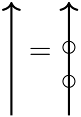
Think: P is induction and Q is restriction on modules for the *Sergeev algebra* (finite Hecke-Clifford algebra).

Morphisms:

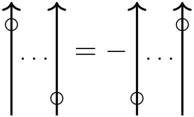


New relations - empty dots

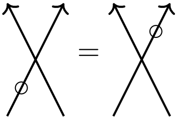
Empty dots correspond to generators c_i of Clifford algebra $\mathcal{C}\ell_n$.
Tr($\mathcal{H}_{tw}$) is $\mathbb{Z}/2\mathbb{Z}$-graded where empty dots have degree 1.



$c_i^2 = 1$



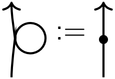
$c_i c_j = -c_j c_i$



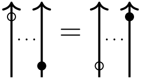
$c_i s_i = s_i c_{i+1}$

Dot interactions

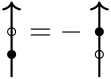
Define:



Empty dots and solid dots on different strands commute.



Empty dots and solid dots on the same strand anticommute.



$$x_i c_i = -c_i x_i$$

Dots, hollow dots, and crossings generate the *degenerate affine Hecke-Clifford algebra* $\mathfrak{H}_n^c$.

$W_{1+\infty}$

$W_{1+\infty}$: unique nontrivial central extension of Lie algebra of differential operators on the circle. Connected to $\mathfrak{gl}_\infty$.

Important in 2D quantum field theory and integrable systems.

[Kac, Wang, Yan, 1998] define a certain subalgebra W^- of $W_{1+\infty}$ fixed by degree-preserving anti-involution.

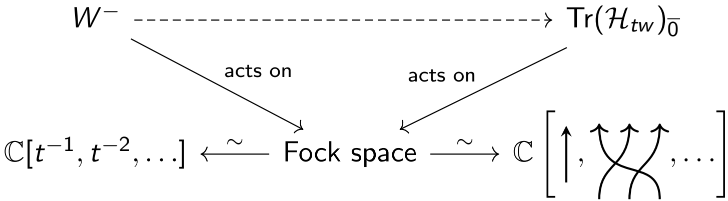
W-algebra W^-

Denote $D = t\partial_t$.

$$W^- = \begin{matrix} \left\{t^j g(D + (j-1)/2) | g \text{ odd polynomial}\right\}, & \text{if } j \text{ even} \\ \left\{t^j g(D + (j-1)/2) | g \text{ even polynomial}\right\}, & \text{if } j \text{ odd} \end{matrix}$$

W^- is generated by t^{-1}, D^3, and $t^{\pm 2}(D \mp 1)$.

Fock space representations

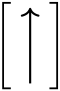


$$\begin{tikzcd}
 & W^{-} \arrow[rr, dashed] \arrow[d, "\text{acts on}"] & & \mathrm{Tr}(\mathcal{H}_{tw})_{\overline{0}} \arrow[dll, "\text{acts on}"'] \\
\mathbb{C}[t^{-1}, t^{-2}, \ldots] & \text{Fock space} \arrow[l, "\sim"'] \arrow[rr, "\sim"] & & \mathbb{C}\left[\uparrow, \ldots\right]
\end{tikzcd}$$

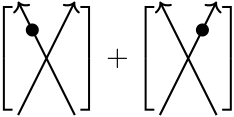
Identify images of generators of each algebra in the Fock space.

Isomorphism

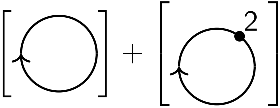
Define an algebra map $\Phi : \mathrm{Tr}(\mathcal{H}_{tw})_{\overline{0}} \to W^-$ by mapping



$$\mapsto \sqrt{2}t^{-1}$$



$$\mapsto -2\sqrt{2}t^2(D \mp 1)$$



$$\mapsto 2D^3$$

and extending algebraically.

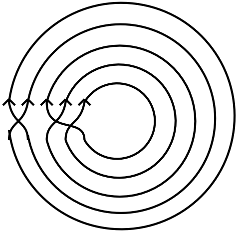
Theorem (Oğuz-Reeks 2017)

The map Φ *is an algebra isomorphism* $\mathrm{Tr}(\mathcal{H}_{tw})_{\overline{0}} \to W^-$.

Centers

The center of a monoidal category $\mathcal{C}$ is the algebra $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$.

The center of $\mathcal{H}_{tw}$ is the algebra of closed diagrams:

$\alpha_{(3,2)} =$ 

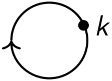
Center of the twisted Heisenberg category

It can be shown that

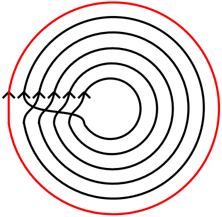
$$Z(\mathcal{H}_{tw}) \cong \mathbb{C}[d_0, d_2, \ldots],$$

where

$d_k :=$

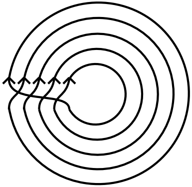


Multiplication is inhomogeneous:

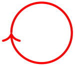


$\alpha_{(5,1)}$

$=$



$\alpha_{(5)}$



$\alpha_{(1)}$

$+$ l. o. t.

The algebra Γ

Let $\Gamma \subset \Lambda$ be the subalgebra of symmetric functions generated by

$$\{p_{2n+1} | n \in \mathbb{N}\}.$$

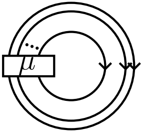
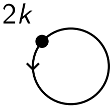
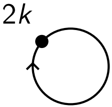
Γ has many interesting bases:

- $\mathfrak{p}_\lambda = p_\lambda +$ l.o.t.
  inhomogenous power sums
- $Q^*_\lambda = Q_\lambda +$ l.o.t.
  factorial Schur Q-functions
- $\mathbf{g}^\uparrow_k, \ \mathbf{g}^\downarrow_{k+1}$
  moments of probability measures on Schur's graph

Center

Theorem (Kvinge, Oğuz, Reeks)

The center $\mathrm{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ *of the twisted Heisenberg category is isomorphic as an algebra to* Γ.

Γ	$\mathfrak{p}_\mu$	$\mathbf{g}_k^\uparrow$	$\mathbf{g}_{k+1}^\downarrow$
diagram in $\mathrm{End}_{\mathcal{H}}(\mathbb{1})$	$\frac{1}{2^{\ell\mu}}$ 		

Selected references

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- [Kac-Wang-Yan] *Quasifinite representations of classical Lie subalgebras of $W_{1+\infty}$*, Adv. Math., 1998.
- [Khovanov] *Heisenberg algebra and a graphical calculus*, Fund. Math., 2010.
- [Kvinge-Oğuz-Reeks] *The center of the twisted Heisenberg category, factorial Schur Q-functions, and transition functions on the Schur graph*, 2017.
- [Oğuz-Reeks] *Trace of the twisted Heisenberg category*, Comm. Math. Phys., 2017.