A Note on $U_q\left(A_{n-1}^{(1)}\right)$-Demazure Crystals

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Motivation

- 1990's: Kashiwara and Lusztig developed crystal base theory
- This theory provides a combinatorial tool to study Lie algebra representation theory
- Applications arise in statistical physics, conformal field theory, differential equations, number theory, combinatorics, and algebraic geometry

Affine Special Linear Lie Algebras

We focus on the affine special linear Lie algebra:

$$A_{n-1}^{(1)} = \hat{sl}(n, \mathbb{C}) = A_{n-1} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

which has the following bracket structure:

$$\begin{aligned}
[x \otimes t^i, y \otimes t^j] &= [x, y] \otimes t^{i+j} + \mathrm{tr}(x, y) i \delta_{i+j,0} c, \\
[d, x \otimes t^i] &= i(x \otimes t^i), \\
[d, c] &= 0.
\end{aligned}$$

Quantum Affine Algebras

Definition

The *quantum affine algebra* $U_q(\mathfrak{g})$ is the associative algebra over $\mathbb{C}(q)$ with unity generated by the elements e_i, f_i, q^h $(h \in \check{P})$ with the following relations:

1. $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$ for $h, h' \in \check{P}$,
2. $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ for $h \in \check{P}$,
3. $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$ for $h \in \check{P}$,
4. $e_i f_j - f_j e_i = \delta_{ij} \frac{q^{s_i h_i} - q^{-s_i h_i}}{q^{s_i} - q^{-s_i}}$,
5. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{s_i}} e_i^{1-a_{ij}-k} e_j e_i^k = 0$, for $i \neq j$,
6. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{s_i}} f_i^{1-a_{ij}-k} f_j f_i^k = 0$, for $i \neq j$.

Crystal Lattices

Definition

Define $A_0 = \left\{ \frac{g(q)}{h(q)} \middle| g(q), h(q) \in \mathbb{C}[q], h(0) \neq 0 \right\}$ to be the principal ideal domain with $\mathbb{C}(q)$ as its field of quotients.

Definition

A free A_0-submodule $\mathcal{L}$ of integrable $\mathfrak{g}$-module V^q is a *crystal lattice* if

1. $\mathbb{C}(q) \otimes_{A_0} \mathcal{L} \cong V^q$,
2. $\mathcal{L} = \bigoplus_{\lambda \in P} \mathcal{L}_\lambda$, $\mathcal{L}_\lambda = \mathcal{L} \cap V^q_\lambda$,
3. $\tilde{e}_i(\mathcal{L}) \subseteq \mathcal{L}$, $\tilde{f}_i(\mathcal{L}) \subseteq \mathcal{L}$ for all $i = 0, 1, \ldots, n$.

Crystal Bases

Definition

A *crystal base* for V^q is a pair $(\mathcal{L}, \mathcal{B})$ such that

1. $\mathcal{L}$ is a crystal lattice,
2. $\mathcal{B}$ is a $\mathbb{C}$-basis of $\mathcal{L}/q\mathcal{L}$,
3. $\mathcal{B} = \cup_{\lambda \in P} \mathcal{B}_\lambda$, $\mathcal{B}_\lambda = \mathcal{B} \cap (\mathcal{L}_\lambda / q\mathcal{L}_\lambda)$,
4. $\tilde{e}_i(\mathcal{B}) \subseteq \mathcal{B} \cup \{0\}$, $\tilde{f}_i(\mathcal{B}) \subseteq \mathcal{B} \cup \{0\}$,
5. For $b, b' \in \mathcal{B}$, $\tilde{f}_i b = b'$ if and only if $\tilde{e}_i b' = b$.

Crystal Graphs

Definition

Given a crystal base $(\mathcal{L}, \mathcal{B})$ for V^q, we can define a *crystal graph* of V^q by letting the elements of $\mathcal{B}$ be the set of vertices and by joining $b \in \mathcal{B}$ to $b' \in \mathcal{B}$ with an i-colored arrow $b \xrightarrow{i} b'$ if and only if $\tilde{f}_i b = b'$.

Perfect Crystals

Although the crystal $B(\lambda)$ for the irreducible highest weight $U_q(A^{(1)}_{n-1})$-module $V^q(\lambda)$ is infinite, it can be realized by a finite crystal called a *perfect crystal*.

Suppose $\lambda(c) = \ell \geq 1$. By [KKMMNN], there exists a finite crystal B_ℓ called a perfect crystal of level ℓ such that

$$B(\lambda) \cong \cdots \otimes B_\ell \otimes B_\ell \otimes B_\ell.$$

For a dominant integral weight λ, consider the unique irreducible integrable highest weight $U_q(A_{n-1}^{(1)})$-module $V^q(\lambda)$.

Definition

For any $w \in \mathcal{W}$, the extremal weight space $V^q(\lambda)_{w\lambda}$ is one-dimensional with basis vector $u_{w\lambda}$, which is called the *extremal vector*.

Definition

For $w \in \mathcal{W}$ and weight space $V^q(\lambda)_{w\lambda} = \mathbb{C}(q)u_{w\lambda}$, the *Demazure module* is $V_w(\lambda) = U_q\left(A_{n-1}^{(1)}\right)^+ u_{w\lambda}$, where $U_q(A_{n-1}^{(1)})^+$ is the subalgebra generated by the e_i's.

Demazure Crystals

Definition

For each $w \in \mathcal{W}$, the Demazure module $V_w(\lambda)$ has a crystal $B_w(\lambda)$, which we call the *Demazure crystal*.

Kashiwara, in 1993, proved that the Demazure crystal is a subset of the crystal for the associated integrable highest weight module [5].

He also showed that the Demazure crystal has the following recursive property:

$$w \prec r_i w \Rightarrow B_{r_i w}(\lambda) = \bigcup_{m \geq 0} \tilde{f}_i^{\,m} B_w(\lambda) \setminus \{0\}.$$

Perfect Crystals

For $A_{n-1}^{(1)}$ and $\ell \geq 1$, consider the perfect crystal:

$$B_\ell = \left\{ (m_1, m_2, \ldots, m_{n-1}, m_0) \in \mathbb{Z}_{\geq 0}^n \middle| \sum_{i=0}^{n-1} m_i = \ell \right\}.$$

The Kashiwara operators act on $b \in B_\ell$ by the following actions:

$$\tilde{f}_0(b) = (m_1 + 1, m_2, \ldots, m_{n-1}, m_0 - 1),$$
$$\tilde{f}_i(b) = (m_1, \ldots, m_i - 1, m_{i+1} + 1, \ldots, m_{n-1}, m_0).$$

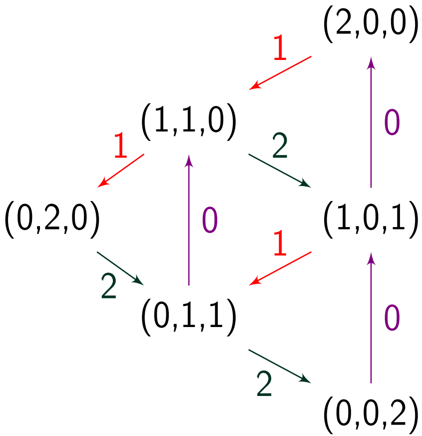
We also define the following:

$$\varphi_i(b) = m_i, \qquad \varphi_0(b) = m_0, \qquad \varphi(b) = \sum_{i=0}^{n-1} \varphi_i(b)\Lambda_i$$

$$\varepsilon_i(b) = m_{i+1}, \qquad \varepsilon_n(b) = m_0, \qquad \varepsilon(b) = \sum_{i=0}^{n-1} \varepsilon_i(b)\Lambda_i.$$

Perfect Crystals

Perfect crystal for $\mathcal{U}_q\left(A_2^{(1)}\right)$ of level 2: $\lambda = 2\Lambda_0$



Path Realizations

Definition

For fixed λ, let b_λ be the unique element of B such that $\varphi(b_\lambda) = \lambda$. Then set

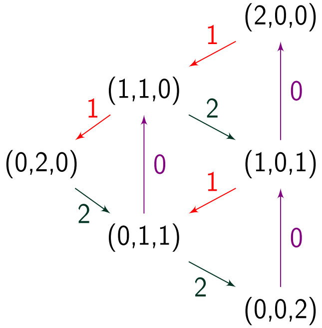
$$\lambda_1 = \lambda, \qquad \lambda_{k+1} = \varepsilon(b_{\lambda_k})$$

$$b_1 = b_\lambda, \qquad b_{k+1} = b_{\lambda_{k+1}}$$

The sequence $p_\lambda = (\cdots \otimes b_{k+1} \otimes \cdots \otimes b_2 \otimes b_1)$ is called the *ground-state path*.

Example:

$\lambda = 2\Lambda_0$



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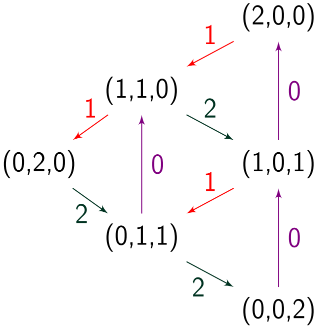

$(0,0,2)$

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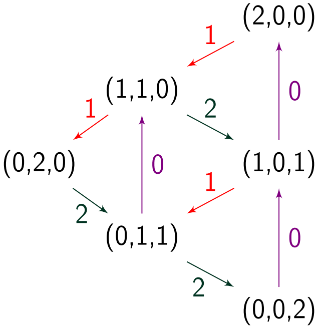

Example: $(0,2,0) \otimes (0,0,2)$

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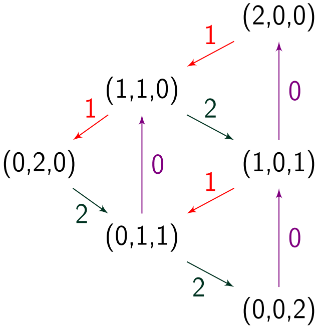

Example: $(2,0,0) \otimes (0,2,0) \otimes (0,0,2)$

Path Realizations

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For fixed λ, let b_λ be the unique element of B such that $\varphi(b_\lambda) = \lambda$. Then set

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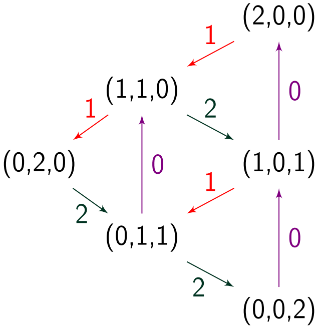

Example: $\cdots \otimes (0,0,2) \otimes (2,0,0) \otimes (0,2,0) \otimes (0,0,2)$

Path Realizations

Definition

A sequence

$$p = (\cdots \otimes p(k+1) \otimes p(k) \otimes \cdots \otimes p(2) \otimes p(1))$$

is called a λ-*path* if $p(k) = b_k$ for $k \gg 1$.

Example: λ-path: $\cdots \otimes (0,0,2) \otimes (2,0,0) \otimes (1,1,0) \otimes (0,1,1)$

We can use λ-paths to find a realization of the affine crystal graph of $V(\lambda)$ and hence for the Demazure crystal.

Main Theorem [Misra, R.]

Theorem

Choose dominant integral weight

$$\lambda = m_0\Lambda_0 + m_1\Lambda_1 + \cdots + m_{n-1}\Lambda_{n-1}.$$

Then

- $\bigcap_{t\geq 0} B_{w(L,t)}(\lambda) = \cdots \otimes b_L \otimes B^{\otimes(L-1)}$
- $\bigcup_{t\geq 0} B_{w(L,t)}(\lambda) = \cdots \otimes b_{L+1} \otimes B^{\otimes(L)}.$

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