Quantum cluster algebra at roots of unity and discriminant formula

Bach Nguyen

Louisiana State University

A conference celebrating the 60-th birthday of Vyjayanthi Chari
June 04, 2018

Quantum Cluster Algebra

We will be working over $\mathbb{Z}[q^{\pm 1/2}]$ for an formal variable q.

- For a skew symmetric matrix $\Gamma \in M_N(\mathbb{Z})$ we define a quantum torus $T_q(\Gamma)$ over $\mathbb{Z}[q^{\pm 1/2}]$.

Quantum Cluster Algebra

We will be working over $\mathbb{Z}[q^{\pm 1/2}]$ for an formal variable q.

- For a skew symmetric matrix $\Gamma \in M_N(\mathbb{Z})$ we define a quantum torus $T_q(\Gamma)$ over $\mathbb{Z}[q^{\pm 1/2}]$.
- A map $M : \mathbb{Z}^N \longrightarrow \mathcal{F}$ is a **toric frame** if there exist Γ such that it defines an embedding $T_q(\Gamma) \hookrightarrow \mathcal{F}$ where $\mathcal{F} \cong Fract(T_q(\Gamma))$.

Quantum Cluster Algebra

We will be working over $\mathbb{Z}[q^{\pm 1/2}]$ for an formal variable q.

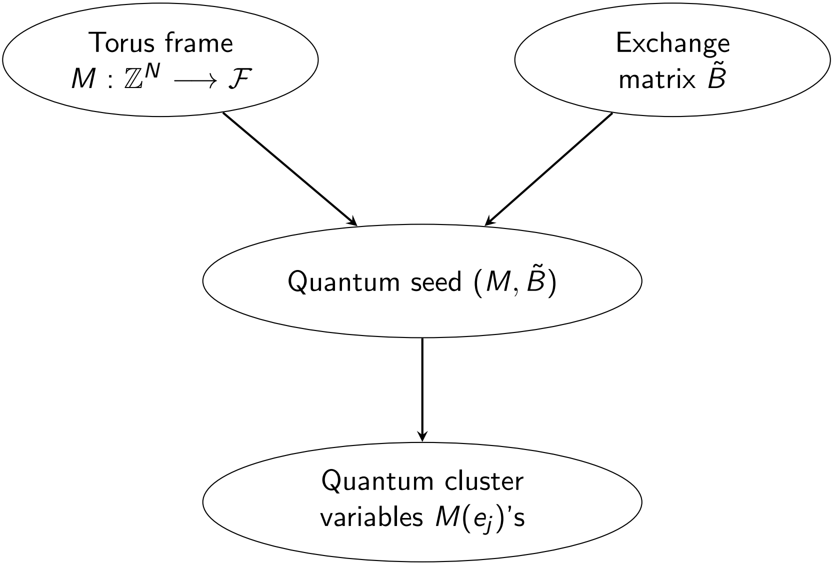
- For a skew symmetric matrix $\Gamma \in M_N(\mathbb{Z})$ we define a quantum torus $T_q(\Gamma)$ over $\mathbb{Z}[q^{\pm 1/2}]$.
- A map $M : \mathbb{Z}^N \longrightarrow \mathcal{F}$ is a **toric frame** if there exist Γ such that it defines an embedding $T_q(\Gamma) \hookrightarrow \mathcal{F}$ where $\mathcal{F} \cong Fract(T_q(\Gamma))$.
- Fix $n \leq N$, and let $\mathbf{ex} \subseteq [1, N]$ such that $|\mathbf{ex}| = n$. An integral matrix $\tilde{B}_{N \times \mathbf{ex}}$ is called **exchange matrix** if the submatrix $B_{\mathbf{ex}}$ is skew symmetrizable.

Quantum Cluster Algebra

We will be working over $\mathbb{Z}[q^{\pm 1/2}]$ for an formal variable q.

- For a skew symmetric matrix $\Gamma \in M_N(\mathbb{Z})$ we define a quantum torus $T_q(\Gamma)$ over $\mathbb{Z}[q^{\pm 1/2}]$.
- A map $M : \mathbb{Z}^N \longrightarrow \mathcal{F}$ is a **toric frame** if there exist Γ such that it defines an embedding $T_q(\Gamma) \hookrightarrow \mathcal{F}$ where $\mathcal{F} \cong Fract(T_q(\Gamma))$.
- Fix $n \leq N$, and let $\mathbf{ex} \subseteq [1, N]$ such that $|\mathbf{ex}| = n$. An integral matrix $\tilde{B}_{N \times \mathbf{ex}}$ is called **exchange matrix** if the submatrix $B_{\mathbf{ex}}$ is skew symmetrizable.
- A compatible pair $(M, \tilde{B})$ is called a **quantum seed** and its corresponding **quantum cluster variables** are $M(e_j)$ for $j \in [1, N]$.

Quantum Cluster Algebra



Quantum Cluster Algebra

- For each $k \in \mathbf{ex}$, one has mutation μ_k which takes quantum seed to quantum seed. $\mu_k((M, \tilde{B})) = (\mu_k(M), \mu_k(\tilde{B}))$.

Quantum Cluster Algebra

- For each $k \in \mathbf{ex}$, one has mutation μ_k which takes quantum seed to quantum seed. $\mu_k((M, \tilde{B})) = (\mu_k(M), \mu_k(\tilde{B}))$.
- The cluster variables indexed by $[1, N] \setminus \mathbf{ex}$ are called **frozen variables**.

Quantum Cluster Algebra

- For each $k \in \mathbf{ex}$, one has mutation μ_k which takes quantum seed to quantum seed. $\mu_k((M, \tilde{B})) = (\mu_k(M), \mu_k(\tilde{B}))$.
- The cluster variables indexed by $[1, N] \setminus \mathbf{ex}$ are called **frozen variables**.
- Let $\mathbf{inv} \subseteq [1, N] \setminus \mathbf{ex}$.

Quantum Cluster Algebra

- For each $k \in \mathbf{ex}$, one has mutation μ_k which takes quantum seed to quantum seed. $\mu_k((M, \tilde{B})) = (\mu_k(M), \mu_k(\tilde{B}))$.
- The cluster variables indexed by $[1, N] \setminus \mathbf{ex}$ are called **frozen variables**.
- Let $\mathbf{inv} \subseteq [1, N] \setminus \mathbf{ex}$.
- Mutation-equivalent of quantum seeds

$$\begin{array}{lcr} (M, \tilde{B}) \xrightarrow{\mu_1} \cdots \xrightarrow{\mu_k} (M', \tilde{B}') \\ M(e_j) & & M'(e_j) \end{array}$$

Quantum Cluster Algebra

- For each $k \in \mathbf{ex}$, one has mutation μ_k which takes quantum seed to quantum seed. $\mu_k((M, \tilde{B})) = (\mu_k(M), \mu_k(\tilde{B}))$.

- The cluster variables indexed by $[1, N] \setminus \mathbf{ex}$ are called **frozen variables**.

- Let $\mathbf{inv} \subseteq [1, N] \setminus \mathbf{ex}$.

- Mutation-equivalent of quantum seeds

$$\begin{array}{lcr} (M, \tilde{B}) \xrightarrow{\mu_1} \cdots \xrightarrow{\mu_k} (M', \tilde{B}') \\ M(e_j) & & M'(e_j) \end{array}$$

- The **quantum cluster algebra** $\mathcal{A}_q(M, \tilde{B}, \mathbf{inv})$ is the algebra generated by all cluster variables $M'(e_j)$, $j \in [1, N]$ and $M'(e_k)^{-1}$, $k \in \mathbf{inv}$ for all quantum seeds $(M', \tilde{B}')$ which are mutation-equivalent to $(M, \tilde{B})$.

Quantum Cluster Algebra at Roots of Unity

Let $\epsilon^{1/2}$ be a primitive ℓ^{th} root of unity and we work over $\mathbb{Z}[\epsilon^{\pm 1/2}]$.

- The based quantum torus is now $T_\epsilon(\Gamma)$

Quantum Cluster Algebra at Roots of Unity

Let $\epsilon^{1/2}$ be a primitive ℓ^{th} root of unity and we work over $\mathbb{Z}[\epsilon^{\pm 1/2}]$.

- The based quantum torus is now $T_\epsilon(\Gamma)$
- Note that we are not specialize $T_q(\Gamma)$ at ϵ but simply define a quantum torus over $\mathbb{Z}[\epsilon^{\pm 1/2}]$.

Quantum Cluster Algebra at Roots of Unity

Let $\epsilon^{1/2}$ be a primitive ℓ^{th} root of unity and we work over $\mathbb{Z}[\epsilon^{\pm 1/2}]$.

- The based quantum torus is now $T_\epsilon(\Gamma)$
- Note that we are not specialize $T_q(\Gamma)$ at ϵ but simply define a quantum torus over $\mathbb{Z}[\epsilon^{\pm 1/2}]$.
- Define the toric frame M as before and Γ be its skew symmetric matrix. Similarly, we have the **root of unity quantum seed** $(M, \tilde{B}, \Gamma)$.

Quantum Cluster Algebra at Roots of Unity

Let $\epsilon^{1/2}$ be a primitive ℓ^{th} root of unity and we work over $\mathbb{Z}[\epsilon^{\pm 1/2}]$.

- The based quantum torus is now $T_\epsilon(\Gamma)$
- Note that we are not specialize $T_q(\Gamma)$ at ϵ but simply define a quantum torus over $\mathbb{Z}[\epsilon^{\pm 1/2}]$.
- Define the toric frame M as before and Γ be its skew symmetric matrix. Similarly, we have the **root of unity quantum seed** $(M, \tilde{B}, \Gamma)$.
- The **quantum cluster algebra at root of unity** $A_\epsilon(M, \tilde{B}, \Gamma, \mathbf{inv})$ is a $\mathbb{Z}[\epsilon^{\pm 1/2}]$-algebra generated by all cluster variables $M'(e_j)$, $j \in [1, N]$ and $M'(e_k)^{-1}$, $k \in \mathbf{inv}$ for all root of unity quantum seeds $(M', \tilde{B}', \Gamma')$ which are mutation-equivalent to $(M, \tilde{B}, \Gamma)$. [N.–Trampel–Yakimov]

Quantum Cluster Algebra at Roots of Unity

Let $A(\tilde{B})$ be the cluster algebra associated to the exchange matrix $\tilde{B}$.

Quantum Cluster Algebra at Roots of Unity

Let $A(\tilde{B})$ be the cluster algebra associated to the exchange matrix $\tilde{B}$.

Theorem 1 (N.–Trampel–Yakimov)

The exchange graphs of $A_{\boldsymbol{q}}(M, \tilde{B})$, $A_{\epsilon}(M, \tilde{B}, \Gamma)$ and $A(\tilde{B})$ are all isomorphic. Moreover, the root of unity quantum cluster algebra satisfies the Laurent phenomenon.

Quantum Cluster Algebra at Roots of Unity

Let $A(\tilde{B})$ be the cluster algebra associated to the exchange matrix $\tilde{B}$.

Theorem 1 (N.–Trampel–Yakimov)

The exchange graphs of $A_{\boldsymbol{q}}(M, \tilde{B})$, $A_\epsilon(M, \tilde{B}, \Gamma)$ *and* $A(\tilde{B})$ *are all isomorphic. Moreover, the root of unity quantum cluster algebra satisfies the Laurent phenomenon.*

Theorem 2 (N.–Trampel–Yakimov)

The elements $M'(e_j)^\ell$, $j \in [1, N]$ *and* $M'(e_k)^{-\ell}$, $k \in \mathbf{inv}$ *are central in* $A_\epsilon(M, \tilde{B}, \Gamma)$. *Moreover, the central subalgebra generated by them is isomorphic to the cluster algebra* $A(\tilde{B})$.

Discriminant of Algebras

Let A be a noncommutative algebra .

- We call (A, tr) is an algebra with trace if $tr : A \longrightarrow A$ such that for any $x, y \in A$

  $$tr(xy) = tr(yx), \quad tr(y)x = xtr(y), \quad tr(xtr(y)) = tr(y)tr(x).$$

  Note that these conditions imply $\mathrm{im}(tr) = C \subset Z(A)$ and tr is C-linear.

Discriminant of Algebras

Let A be a noncommutative algebra .

- We call (A, tr) is an algebra with trace if $tr : A \longrightarrow A$ such that for any $x, y \in A$

  $$tr(xy) = tr(yx), \quad tr(y)x = xtr(y), \quad tr(xtr(y)) = tr(y)tr(x).$$

  Note that these conditions imply $\mathrm{im}(tr) = C \subset Z(A)$ and tr is C-linear.
- Let (A, tr) be an algebra with trace and $Y = \{y_1, ..., y_n\} \subset A$. We define discriminant of Y to be

  $$d(Y : tr) = \det[tr(y_i y_j)] \in C.$$

Discriminant of Algebras

Let A be a noncommutative algebra .

- We call (A, tr) is an algebra with trace if $tr : A \longrightarrow A$ such that for any $x, y \in A$

  $$tr(xy) = tr(yx), \quad tr(y)x = xtr(y), \quad tr(xtr(y)) = tr(y)tr(x).$$

  Note that these conditions imply $\text{im}(tr) = C \subset Z(A)$ and tr is C-linear.
- Let (A, tr) be an algebra with trace and $Y = \{y_1, ..., y_n\} \subset A$. We define discriminant of Y to be

  $$d(Y : tr) = \det[tr(y_i y_j)] \in C.$$

- When A is free of rank n over some central subalgebra C, we use the map $tr : A \hookrightarrow M_n(C) \longrightarrow C$. Then discriminant of A over C is $d(A/C) =_{C^\times} d(Y : tr)$ for a chosen C-basis Y of A.

Discriminant of Quantum Cluster Algebra

Suppose Θ is a finite set of seeds in $A_\epsilon(M, \tilde{B}, \Gamma)$ such that every 2 seeds in Θ are connected by a sequence of mutations in Θ and every nonfrozen vertex is mutated at least one time in Θ.

Discriminant of Quantum Cluster Algebra

Suppose Θ is a finite set of seeds in $A_\epsilon(M, \tilde{B}, \Gamma)$ such that every 2 seeds in Θ are connected by a sequence of mutations in Θ and every nonfrozen vertex is mutated at least one time in Θ.

Proposition 3 (N.–Trampel–Yakimov)

Let $A_\epsilon(\Theta)$ be the subalgebra of $A_\epsilon(M, \tilde{B}, \Gamma)$ generated by the cluster variables in Θ. Let $C_\epsilon(\Theta)$ be the central subalgebra of $A_\epsilon(\Theta)$ generated by the ℓ^{th} power of the cluster variables. Then $A_\epsilon(\Theta)$ is finitely generated as a $C_\epsilon(\Theta)$-module.

Discriminant of Quantum Cluster Algebra

Theorem 4 (N.–Trampel–Yakimov)

Suppose $A_\epsilon(\Theta)$ is free over $C_\epsilon(\Theta)$. Then

$$d(A_\epsilon(\Theta)/C_\epsilon(\Theta)) = \prod(\textit{noninverted frozen variables})$$

.

Discriminant of Quantum Cluster Algebra

Theorem 4 (N.–Trampel–Yakimov)

Suppose $A_\epsilon(\Theta)$ *is free over* $C_\epsilon(\Theta)$. *Then*

$$d(A_\epsilon(\Theta)/C_\epsilon(\Theta)) = \prod(\textit{noninverted frozen variables})$$

Consider the quantum group $U_q(\mathfrak{g})$ for a symmetrizable Kac–Moody algebra $\mathfrak{g}$, then the quantum Schubert cell algebra $U_q(\mathfrak{n}_+ \cap w(\mathfrak{n}_-))$ is a cluster algebra due to [Geiss–Leclerc–Schroer, Goodearl–Yakimov].

Discriminant of Quantum Cluster Algebra

Theorem 4 (N.–Trampel–Yakimov)

Suppose $A_\epsilon(\Theta)$ is free over $C_\epsilon(\Theta)$. Then

$$d(A_\epsilon(\Theta)/C_\epsilon(\Theta)) = \prod(\textit{noninverted frozen variables})$$

Consider the quantum group $U_q(\mathfrak{g})$ for a symmetrizable Kac–Moody algebra $\mathfrak{g}$, then the quantum Schubert cell algebra $U_q(\mathfrak{n}_+ \cap w(\mathfrak{n}_-))$ is a cluster algebra due to [Geiss–Leclerc–Schroer, Goodearl–Yakimov].

Theorem 5 (N.–Trampel–Yakimov)

For any symmetrizable Kac–Moody algebra $\mathfrak{g}$, $w \in W$, and ϵ an odd primitive root of unity,

$$d(U_\epsilon(\mathfrak{n}_+ \cap w(\mathfrak{n}_-))/C_\epsilon) = \prod \Delta_{\omega_i, w\omega_i}^{\ell^{t+1}(\ell-1)}$$

where t is the length of w.