Graded Representations of Current Algebras

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Interactions of Quantum Affine Algebras with Cluster Algebras, Current Algebras and Categorification

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Current Algebra

Definition

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. The current algebra $\mathfrak{g}[t]$ is the vector space $\mathfrak{g} \otimes \mathbb{C}[t]$ with bracket

$$[a \otimes f, b \otimes g] = [a, b] \otimes fg \quad \forall a, b \in \mathfrak{g},\ f, g \in \mathbb{C}[t].$$

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$\mathrm{ev}_z V$ is an irreducible $\mathfrak{g}[t]$-module if and only if V is an irreducible $\mathfrak{g}$-module.

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The only $\mathbb{Z}$-graded evaluation modules occur when $z = 0$.

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Then, it is a simple exercise to show that $ev_0 V(n)$ are the graded irreducible representations of $\mathfrak{sl}_2[t]$.

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Then, define

$$y(r,s) = \sum_{(b_p)_{p\geq 0} \in S(r,s)} y_0^{(b_0)} y_1^{(b_1)} \cdots y_s^{(b_s)}$$

where $y_j^{(p)} := \frac{y_j^p}{p!}$.

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$V(\xi)$ modules

Definition (Chari—Venkatesh, E. Feigin)

Let $\xi = (\xi_1 \geq \cdots \geq \xi_m > 0)$ be a partition. Then, $V(\xi)$ is the $\mathfrak{sl}_2[t]$-module generated by v_ξ with defining relations:

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$$x_r.v_\xi = 0$$

$$h_r.v_\xi = |\xi|\delta_{r,0}v_\xi, \quad \text{where } |\xi| = \sum_{j\geq 1}\xi_j$$

$$y_0^{|\xi|+1}v_\xi = 0$$

$$y(r,s)v_\xi = 0, \text{where } s + r \geq 1 + rk + \sum_{j\geq k+1}\xi_j \text{ for some } k \in \mathbb{Z}_+.$$

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If $\xi = \ell_2^{m_2}\ell_1$, then $V(\xi)$ is a Demazure module.

Motivation

The motivation for finding a new presentation comes from the relations

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1. The relationship between k and the pair r and s is unknown in general.
2. Notice that s only appears on the left hand side of the inequality.

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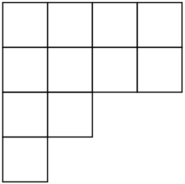
$$\begin{aligned} \nu_1 &= 4 \\ \nu_2 &= 3 \\ \nu_3 &= 2 \\ \nu_4 &= 2 \\ \nu_j &= 0 \quad \forall j \geq 5 \end{aligned}$$

The Young diagram of $\xi = 4^2 21$ is



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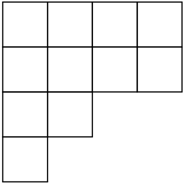

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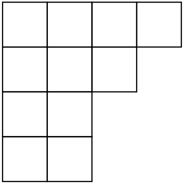

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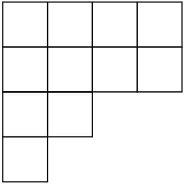

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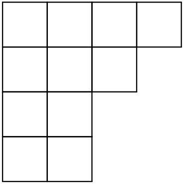

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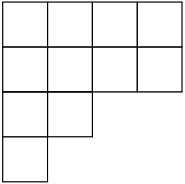

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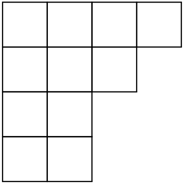

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For $j \in \mathbb{N}$, we define a new partition $\xi^{(j)}$ given by

$$\xi^{(j)} := (\xi_1 \geq \xi_2 \geq \cdots \geq \xi_j \geq 0).$$

A Presentation of ξ using ξ^{tr}

Theorem (M.)

The module $V(\xi)$ is isomorphic to the quotient of the local Weyl module $W_{loc}(|\xi|)$ by the $\mathfrak{sl}_2[t]$–submodule generated by the elements

$$\{y(r, |(\xi^{tr})^{(r)}| - r + 1)w_{|\xi|} : \quad 1 \leq r \leq \xi_1 - 1\}.$$

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As a consequence of our proof, we have shown in $V(\xi)$ we have the relation

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This shows that $V(\xi)$ is finitely presented as a quotient of a local Weyl module.

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- When is $\mathrm{Hom}(V(\xi), V(\xi')) \neq 0$?
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Define $\xi^+ = (\xi_1^+ \geq \xi_2^+ \geq \cdots \geq \xi_{m-1}^+ \geq \xi_m^+ \geq 0)$ is the unique partition associated to the n-tuple $(\xi_1, \xi_2, \ldots, \xi_{m-2}, \xi_{m-1} + 1, \xi_m - 1)$.

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Theorem (Chari—Venkatesh, 2014)

For $m > 1$, there exists a short exact sequence of $\mathfrak{sl}_2[t]$-modules

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With our new presentation, we provided a different construction of φ^-.

Construction of φ^-

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Lemma (M.)

There exists a well defined map of $\mathfrak{sl}_2[t]-$modules $V(\tilde{\xi}_1) \to V(\xi)$ extending the assignment $v_{\tilde{\xi}_1} \to y_{\nu_1(\xi)-1} v_\xi$.

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Then, φ^- is a composition of the well defined maps from this lemma.

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We also use them to study the tensor products of local Weyl modules and produce filtrations by Demazure modules.