Global Weyl modules and maximal parabolics of twisted affine Lie algebras

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Interactions of quantm affine algebras with cluster algebras, current algebras and categorification June 5, 2018

- For a simple Lie algebra $\mathfrak{g} \supset \mathfrak{h}$
- $\Delta = \{\alpha_i : i \in I\}$
- $\Phi^+ = \{\sum_{i \in I} a_i \alpha_i : a_i \geq 0 \, \forall i\}$
- $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \mathfrak{n}^+$
- P^+ dominant integral weights

Universal highest modules

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s.s.	• Verma Module	

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Affine Lie algebras

Proposition (Chari–Pressley, 2001)

- *Let V be an integrable $U(\mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}d)$-module generated by a non-zero element $v \in V_\lambda^+$. Then V is a quotient of $W(\lambda)$, the global Weyl module.*

- $W(\lambda)$ is a $(U(\mathfrak{g}[t,t^{-1}]), \mathbf{A}_\lambda)$-bimodule
  - $\mathbf{A}_\lambda = U(\mathfrak{h}[t,t^{-1}])/Ann_{U(\mathfrak{h}[t,t^{-1}])} w_\lambda$

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Maximal parabolic

- $\mathfrak{g} = \bigoplus_{s=0}^{k-1} \mathfrak{g}_s$, $\mathfrak{g}_0$ is simple, and each $\mathfrak{g}_s$, $1 \leq s \leq k-1$, is an irreducible $\mathfrak{g}_0$-module

k	$\mathfrak{g}$	$\mathfrak{g}_0$	$\mathfrak{g}_k$
2	A_{2n}	B_n	$V_{\mathfrak{g}_0}(2\theta_0^s)$
2	A_{2n-1}, $n \geq 2$	C_n	$V_{\mathfrak{g}_0}(\theta_0^s)$
2	D_{n+1}, $n \geq 3$	B_n	$V_{\mathfrak{g}_0}(\theta_0^s)$
2	E_6	F_4	$V_{\mathfrak{g}_0}(\theta_0^s)$
3	D_4	G_2	$V_{\mathfrak{g}_0}(\theta_0^s)$

- $\sigma : \mathbb{C}[t, t^{-1}] \to \mathbb{C}[t, t{-}1]$ by $\sigma(f(t)) = f(\xi^{-1} t)$, ξ a k-th root of unity

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Realization

- The Maximal parabolic,
  $\mathfrak{p}_j =< x_i^{\pm} \otimes 1, x_0^{\pm} \otimes t^{\pm 1}, x_j^{+} \otimes 1 > \subset (\mathfrak{g}[t, t^{-1}])^{\sigma}$

Proposition (L.)

For a simply laced Lie algebra $\mathfrak{g}$ and some $0 < j \leq n$ $\mathfrak{p}_j \simeq (\mathfrak{g}[t]^{\sigma})^{\tau}$ for some automorphism τ.

- Since the fixed points of $\mathfrak{g}^{\sigma\tau}$ form a semisimple Lie algebra, we can define I_0, Δ_0, and P_0^+.

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Global Weyl Module

- For $\lambda \in P_0^+$, $W(\lambda)$ is generated by w_λ with relations:

$$h.w_\lambda = \lambda(h)w_\lambda \quad \mathfrak{n}^+[t]^{\sigma\tau}.w_\lambda = 0, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1}.w_\lambda = 0.$$

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- To obtain a better description of $W(\lambda)$ we need to describe $\mathbf{A}_\lambda$

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Theorem (L.)

$\mathbf{A}_\lambda / \operatorname{Jac}(\mathbf{A}_\lambda) \simeq$
$\mathbb{C}[P_{i,r_i} : r_i \leq \min\{\lambda(h_i), \lambda(h_0)\}]/$
$< P_{1,r_1} \cdots P_{n,r_n} : \sum_{i \in I_0} \mathbf{a}_i(\alpha_0) r_i \geq \lambda(h_0) + 1 >$

- If $\mathbf{a}_i(_0) \leq 1 \; \forall i \in I_0$ then $\operatorname{Jac}(\mathbf{A}_\lambda) = 0$.

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- If $\mathbf{a}_i(_0) \leq 1 \ \forall i \in I_0$ then $\operatorname{Jac}(\mathbf{A}_\lambda) = 0$.

- $W(\lambda)$ is irreducible iff $\{i \in I_0 : \lambda(h_i) > 0\} \cup \{i \in I_0 : \mathbf{a}_i(\alpha_0) = \mathbf{a}_i(\theta_{k\mathbf{a}_j(\alpha_0)} - 1)\}$

- The following are equivalent:
  1. $W(\lambda)$ is finite-dimensional
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Future Work

- Repeat for Yangians

Thank you for your time.

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