Global Weyl modules and maximal parabolics of twisted affine Lie algebras

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University of California, Riverside

Interactions of quantum affine algebras with cluster algebras, current algebras and categorification June 5, 2018
For a simple Lie algebra $g \supset h$

$\Delta = \{ \alpha_i : i \in I \}$

$\Phi^+ = \{ \sum_{i \in I} a_i \alpha_i : a_i \geq 0 \ \forall i \}$

$b = h \oplus \bigoplus_{\alpha \in \Phi^+} g_\alpha = h \oplus n^+$

$P^+$ dominant integral weights
### Universal highest modules

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<td>Verma Module</td>
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- $\mathfrak{g}$ or $\hat{\mathfrak{g}}$ represents the Lie algebra or its universal enveloping algebra
- $\mathfrak{p}$ parabolic indicates a parabolic subalgebra
**Universal highest modules**

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Proposition (Chari–Pressley, 2001)

Let $V$ be an integrable $U(\mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}d)$-module generated by a non-zero element $v \in V^+_\lambda$. Then $V$ is a quotient of $W(\lambda)$, the global Weyl module.

- $W(\lambda)$ is a $(U(\mathfrak{g}[t, t^{-1}]), A_\lambda)$-bimodule
- $A_\lambda = U(\mathfrak{h}[t, t^{-1}])/ \text{Ann}_{U(\mathfrak{h}[t, t^{-1}])} w_\lambda$
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| affine  | Global Weyl module (untwisted and twisted) | (Untwisted) Global Weyl Module (Twisted)??
Maximal parabolic

\[ g = \bigoplus_{s=0}^{k-1} g_s, \ g_0 \text{ is simple, and each } g_s, 1 \leq s \leq k - 1, \text{ is an irreducible } g_0\text{-module} \]

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\[ \sigma : \mathbb{C}[t, t^{-1}] \rightarrow \mathbb{C}[t, t^{-1}] \text{ by } \sigma(f(t)) = f(\xi^{-1}t), \ \xi \text{ a } k\text{-th root of unity} \]
Maximal parabolic

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The Maximal parabolic, 
\[ p_j = \langle x_i^{\pm} \otimes 1, x_0^{\pm} \otimes t^{\pm 1}, x_j^{+} \otimes 1 > \subset (g[t, t^{-1}])^\sigma \]

Proposition (L.)

For a simply laced Lie algebra \( g \) and some \( 0 < j \leq n \)

\[ p_j \simeq (g[t]^\sigma)^\tau \text{ for some automorphism } \tau. \]

Since the fixed points of \( g^{\sigma \tau} \) form a semisimple Lie algebra, we can define \( I_0, \Delta_0, \) and \( P_0^+ \).
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Realization

- The Maximal parabolic,
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Global Weyl Module

- For $\lambda \in P_0^+$, $W(\lambda)$ is generated by $w_\lambda$ with relations:

  $h.w_\lambda = \lambda(h)w_\lambda \quad n^+[t]^{\sigma^\tau}.w_\lambda = 0, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1}.w_\lambda = 0$.

- For $\lambda \in P_0^+$, $W(\lambda)$ is a $(U(g[t]^{\sigma^\tau}), A_\lambda)$-bimodule.

- $A_\lambda = U(\mathfrak{h}[t]^{\sigma^\tau})/\text{Ann}_{U(\mathfrak{h}[t]^{\sigma^\tau})}W_\lambda$

- To obtain a better description of $W(\lambda)$ we need to describe $A_\lambda$
Motivation
Background
Realization of Maximal Parabolic
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Global Weyl Module

For $\lambda \in P_0^+$, $W(\lambda)$ is generated by $w_\lambda$ with relations:

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- To obtain a better description of $W(\lambda)$ we need to describe $A_\lambda$.
Theorem (L.)

\[ \mathbb{A}_\lambda \mathbb{A}_\lambda \left/ \text{Jac}(\mathbb{A}_\lambda) \right. \simeq \mathbb{C}[P_{i,r_i} : r_i \leq \min\{\lambda(h_i), \lambda(h_0)\}] / \left< P_{1,r_1} \cdots P_{n,r_n} : \sum_{i \in I_0} a_i(\alpha_0) r_i \geq \lambda(h_0) + 1 \right> \]

• If \( a_i(0) \leq 1 \ \forall i \in I_0 \) then \( \text{Jac}(\mathbb{A}_\lambda) = 0 \).
Theorem (L.)

\[ \mathbb{A}_\lambda / \text{Jac}(\mathbb{A}_\lambda) \cong \mathbb{C}[P_{i,r_i} : r_i \leq \min\{\lambda(h_i), \lambda(h_0)\}] / \]
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$A_{\lambda}$

- $W(\lambda)$ is irreducible iff
  \[ \{ i \in l_0 : \lambda(h_i) > 0 \} \cup \{ i \in l_0 : a_i(\alpha_0) = a_i(\theta_k a_j(\alpha_0) - 1) \} \]

- The following are equivalent:
  1. $W(\lambda)$ is finite-dimensional
  2. $A_{\lambda}$ is finite-dimensional
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Future Work

- Repeat for Yangians
Thank you for your time.
References

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