# Heisenberg categories, towers of algebras, and symmetric functions

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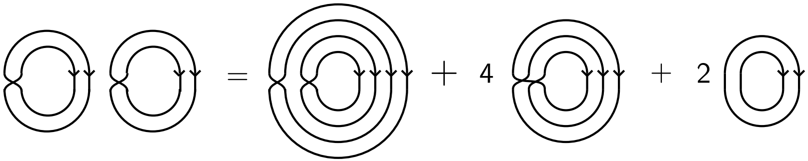
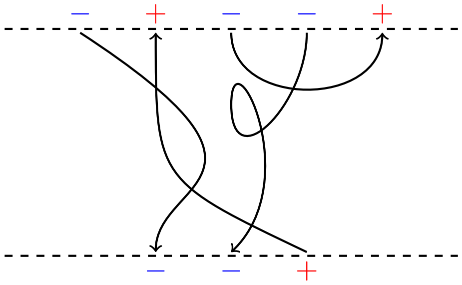
QAA Conference 2018



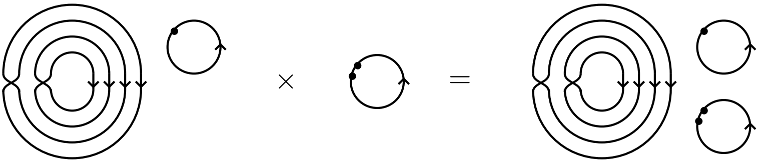
Khovanov's Heisenberg category $\mathcal{H}$ is the Heisenberg category "built" from induction/restriction functors between symmetric group algebras.

- There is a categorical action of $\mathcal{H}$ on $\bigoplus_{n\geq 0} \mathbb{C}[S_n]$-mod.
- This action gives a surjective map from $Z(\mathcal{H})$ to $Z(\mathbb{C}[S_n])$ for any $n \geq 0$.



# Center of $\mathcal{H}$

By definition the center $Z(\mathcal{H})$ of $\mathcal{H}$ is graphically the commutative $\mathbb{C}$-algebra of all closed diagrams.



$\mathcal{H}$ is **rich** in representation-theoretic data (morphism spaces contain all symmetric groups, affine degenerate Hecke algebras).

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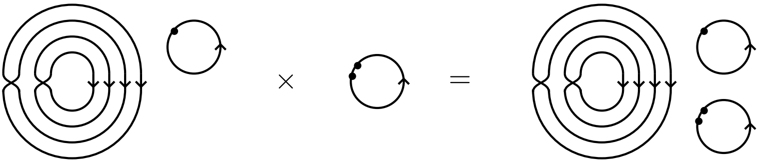


$\mathcal{H}$ is **rich** in representation-theoretic data (morphism spaces contain all symmetric groups, affine degenerate Hecke algebras).

$$\Downarrow$$

$Z(\mathcal{H})$ should contain interesting information.

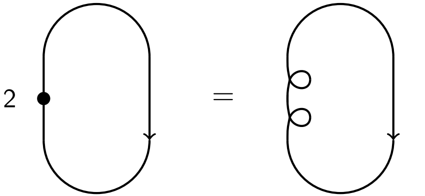
# Center of $\mathcal{H}$

**Theorem (Khovanov)**

$$Z(\mathcal{H}) \cong \mathbb{C}[c_0, c_1, \dots] \cong \mathbb{C}[\tilde{c}_2, \tilde{c}_3, \dots]$$

*where*


In our notation for $\mathcal{H}$ a dot labelled with a $k$ is $k$ right-twisted curls.



The shifted symmetric functions $\Lambda^*$ are a deformation of the classical symmetric functions that can be realized either as:

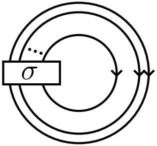
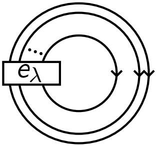
- An analogue of symmetric functions which are "shifted symmetric", i.e. symmetric in the new variables $x_i' = x_i - i$ for all $i \geq 1$.
- Certain functions from Young diagrams to $\mathbb{C}$.

**Theorem (K., Licata, Mitchell)**

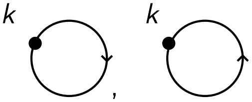
*As an algebra $Z(\mathcal{H})$ is isomorphic (in a diagrammatically meaningful way) to the shifted symmetric functions $\Lambda^*$,*

$$Z(\mathcal{H}) \cong \Lambda^*.$$

# Describing $\mathcal{H}$

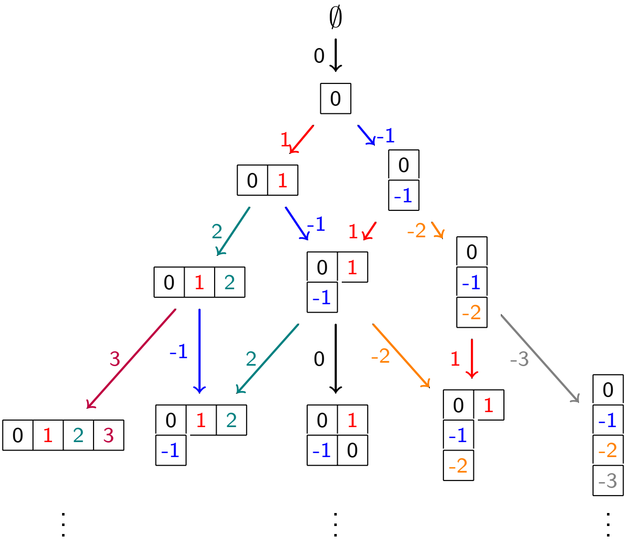
| | | Shifted symmetric function |
|---|---|---|
| $\sigma \in \mathbb{C}[S_n]$,   | $\longleftrightarrow$ | Power sum analogue which evaluates to normalized characters of $S_n$: $p^{\#}_{\lambda}$ |
| idempotent $e_\lambda \in \mathbb{C}[S_n]$,   | $\longleftrightarrow$ | Shifted Schur function $s^*_\lambda$ |

**Remaining question:**



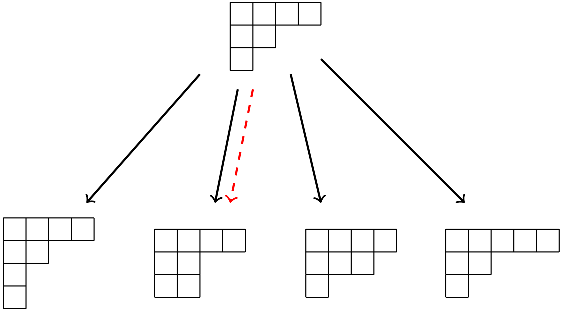

$$\longmapsto \quad ? \in \Lambda^*$$

# Young's Lattice



# Motivation for transition measure

Assign probability of moving from $\lambda = (4, 2, 1)$ to $\mu = (4, 2, 2)$?



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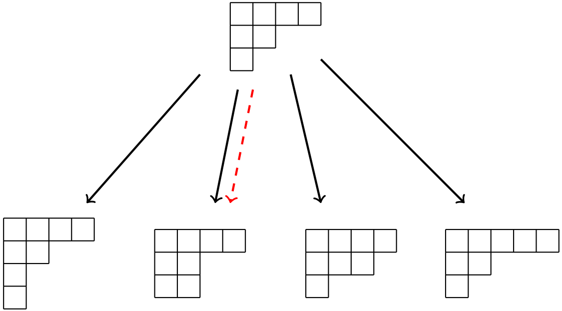

One choice is the *transition probability*: $\widehat{q}_\lambda((4,2,2)) = \frac{\dim(L^{(4,2,2)})}{|\mu| \dim(L^\lambda)}$

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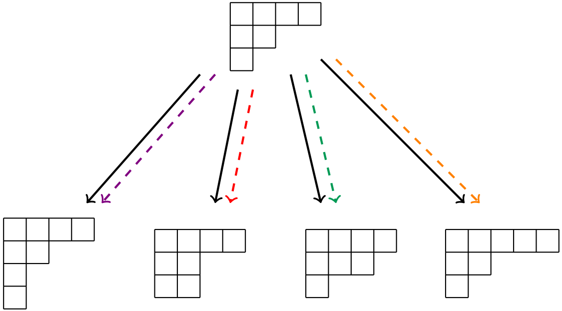

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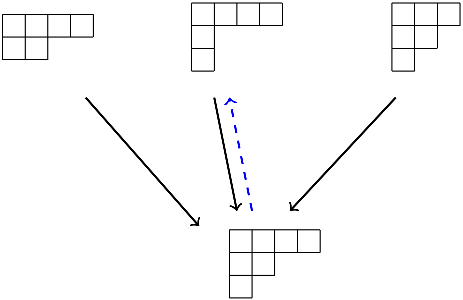
$$\frac{\dim(L^{(4,2,1,1)})}{|\mu| \dim(L^\lambda)} + \frac{\dim(L^{(4,2,2)})}{|\mu| \dim(L^\lambda)} + \frac{\dim(L^{(4,3,1)})}{|\mu| \dim(L^\lambda)} + \frac{\dim(L^{(5,2,1)})}{|\mu| \dim(L^\lambda)} = 1$$

# Motivation for co-transition measure

Dually, assign probability of moving from $\lambda = (4, 2, 1)$ to $\mu = (4, 1, 1)$?



*Co-transition probability*: $\check{q}_\lambda((4,1,1)) = \frac{\dim(L^{(4,1,1)})}{\dim(L^\lambda)}$

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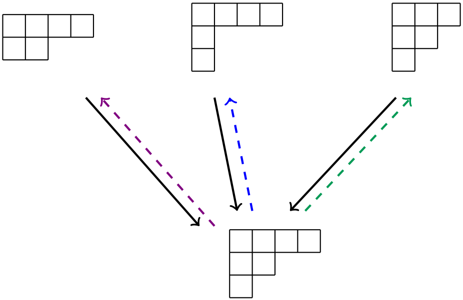


*Co-transition probability*: $\check{q}_\lambda((4,1,1)) = \frac{\dim(L^{(4,1,1)})}{\dim(L^\lambda)}$

$$\frac{\dim(L^{(4,2)})}{\dim(L^\lambda)} + \frac{\dim(L^{(4,1,1)})}{\dim(L^\lambda)} + \frac{\dim(L^{(3,2,1)})}{\dim(L^\lambda)} = 1$$

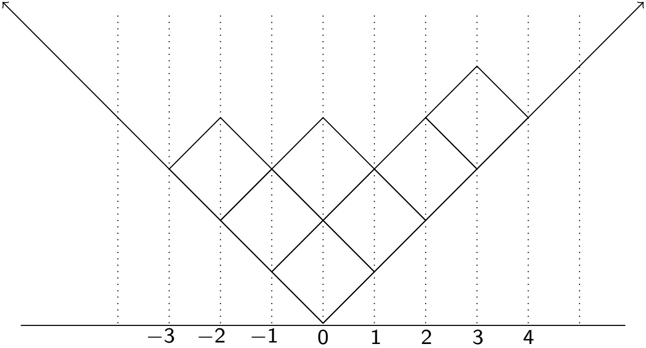
# The transition/co-transition measure

For each $\lambda$, Kerov constructed two probability measures on $\mathbb{R}$ based on the transition/co-transition measure for $\lambda$:

$$\widehat{\omega}_\lambda = \textit{ transition measure for } \lambda = \sum_{\lambda \nearrow \mu} \widehat{q}_\lambda(\mu) \delta_{\text{content}(\mu/\lambda)}$$

$$\widecheck{\omega}_\lambda = \textit{ co-transition measure for } \lambda = \sum_{\mu \nearrow \lambda} \widecheck{q}_\lambda(\mu) \delta_{\text{content}(\lambda/\mu)}$$

**Example:** $\lambda = (4, 2, 1)$.



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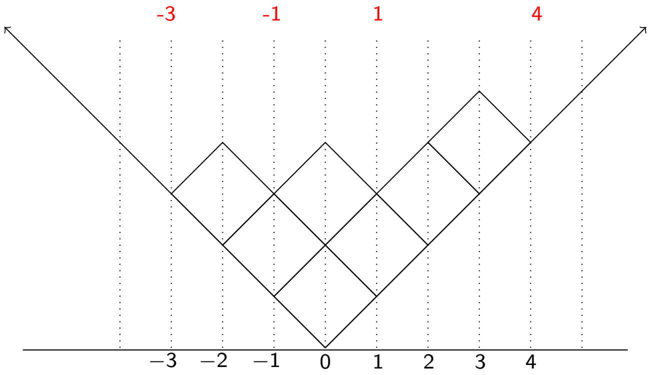

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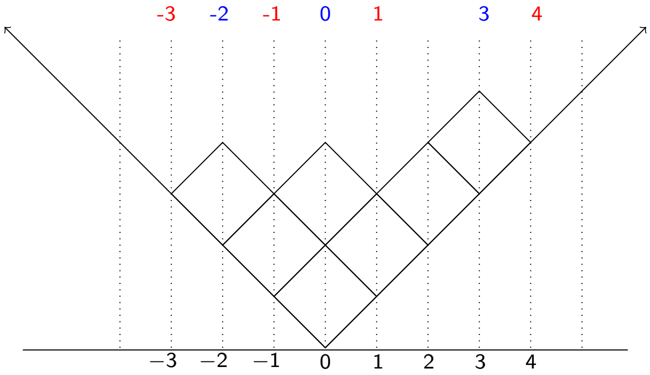
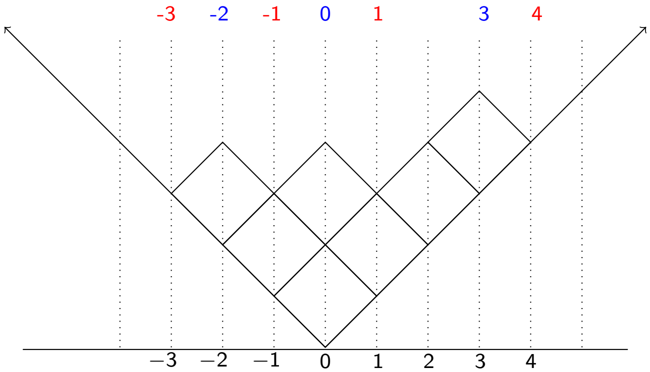


**Example:** $\lambda = (4, 2, 1)$.

$$\widehat{\omega}_\lambda = \quad \widehat{q}_\lambda((4,2,1,1))\delta_{-3} + \widehat{q}_\lambda((4,2,2))\delta_{-1} + \widehat{q}_\lambda((4,3,1))\delta_{1} + \widehat{q}_\lambda((5,2,1))\delta_{4}$$



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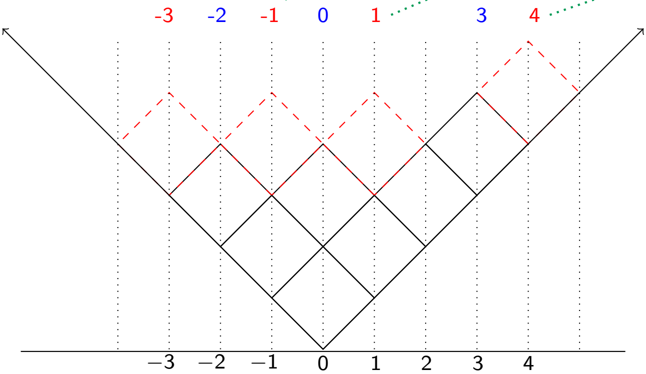
Contents of cells we can add



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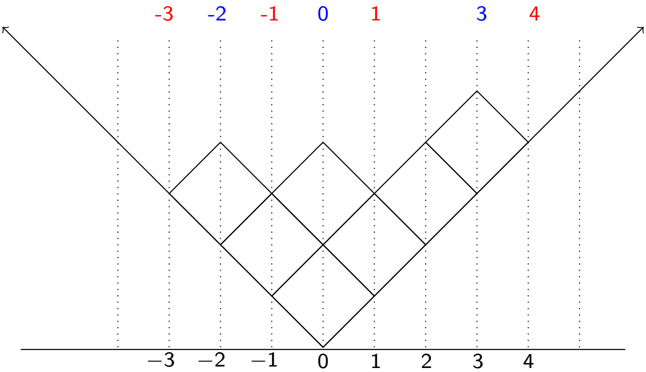
$$\breve{\omega}_\lambda = \breve{q}_\lambda((4,2))\delta_{-2} + \breve{q}_\lambda((4,1,1))\delta_0 + \breve{q}_\lambda((3,2,1))\delta_3$$



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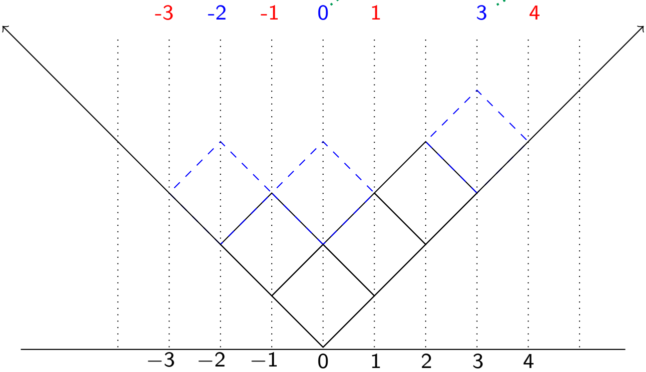
Contents of cells we can remove



# Moments of Kerov's (co)-transition measure

There are moments for probability measures $\widehat{\omega}_\lambda$ and $\widecheck{\omega}_\lambda$.

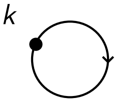
- $\widehat{m}_k(\lambda) = k$th moment of $\widehat{\omega}_\lambda$,
- $\widecheck{m}_k(\lambda) = k$th moment of $\widecheck{\omega}_\lambda$.

In fact, we can view $\widehat{m}_k$, $\widecheck{m}_k$ as functions on $\mathcal{P}$ by

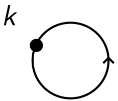
$$\lambda \xmapsto{\widehat{m}_k} \widehat{m}_k(\lambda), \qquad \lambda \xmapsto{\widecheck{m}_k} \widecheck{m}_k(\lambda).$$

These functions encode data related to dimensions of simple $\mathbb{C}[S_n]$-modules for $n \geq 0$. Furthermore, $\widehat{m}_k$, $p_1^{\#}\widecheck{m}_k \in \Lambda^*$ (Lassalle).

Then




$$\longmapsto \quad p_1^{\#}\check{m}_k \in \Lambda^*$$




$$\longmapsto \quad \widehat{m}_k \in \Lambda^*$$

# Interpretation

So the diagrammatics of bubbles encode the information for pairs of Markov transition kernels up and down levels in Young's graph.

This framework also appears in:

- The study of the limit shape of Young diagrams under the Plancherel growth process (Kerov-Vershik).
- The generation of infinite dimensional diffusion processes (Borodin-Olshanski).

That these probabilistic structures arise in Heisenberg categories is not a coincidence, it seems to be another indication of the planar nature of free probability.

# Describing $\mathcal{H}$

Examples: known and in progress.

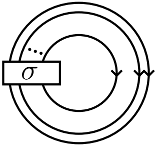
| Heisenberg category | Tower of algebras | $Z(\mathcal{H}_A)$ |
|---|---|---|
| Khovanov's Heisenberg category | symmetric groups $\{S_n\}$ | shifted symmetric functions, Farahat-Higman algebra, (K.-Licata-Mitchell) |
| spin Heisenberg category (Cautis-Sussan) | Sergeev algebras $\{\mathbb{S}_n\}$ | $\mathbb{C}[p_1, p_3, p_5, \ldots] \subset \Lambda$ (K.-Oğuz-Reeks) |
| $q$-Heisenberg categories (Licata-Savage) | Hecke algebras $\{H_n\}$ | *in progress* |
| higher Heisenberg categories (Mackaay-Savage) | degenerate cyclotomic Hecke algebras $\{\bar{H}_n^\lambda\}$ | *in progress* |

# Describing $\mathcal{H}$

The pattern seems to be...

**Symmetric function (or analogue )**

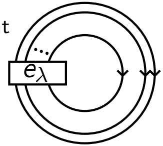
$\sigma \in A_n$,




$\longleftrightarrow$

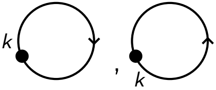
Function which evaluates to normalized "characters" of $A_n$: $p_\lambda^\#$, class symmetric function, ...

idempotent $e_\lambda \in A_n$,




$\longleftrightarrow$

Factorial Schur function analogue for $\{A_n\}$: shifted Schur function, factorial $Q$-Schur function, ...




$\longleftrightarrow$

Encodes Markov transition probabilities defined by $\mathrm{Ind}_{A_n}^{A_{n+1}} \uparrow$, $\mathrm{Res}_{A_n}^{A_{n+1}} \downarrow$: moments of Kerov's (co)-transition measure, Petrov's up/down functions on strict partitions, ...

Thank you.