A path approach to Kostant modules

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Joint work with

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Notations

- $\mathfrak{g}$: symmetrizable Kac-Moody algebra.
- $\mathfrak{h}$: Cartan subalgebra.
- $\mathfrak{b}$: Borel subalgebra, containing $\mathfrak{h}$.
- W : Weyl group.
- Λ : integral weight lattice.
- Λ^+ : dominant integral weight lattice.
- $V(\lambda)$: irreducible integrable representation with highest weight $\lambda \in \Lambda^+$.

Kostant Modules

Fix λ, μ dominant integral weights (throughout talk) and w an element of the Weyl group. Let v_λ be a highest weight vector in $V(\lambda)$. Let $v_{w\mu}$ be a non-zero vector in the (one-dimensional) weight space of weight $w\mu$ in the irreducible representation $V(\mu)$. The *Kostant module* $K(\lambda, w, \mu)$ is defined to be the cyclic submodule of the tensor product $V(\lambda) \otimes V(\mu)$ generated by the element $v_\lambda \otimes v_{w\mu}$:

$$K(\lambda, w, \mu) := U\mathfrak{g}(v_\lambda \otimes v_{w\mu})$$

where $U\mathfrak{g}$ denotes the universal enveloping algebra of $\mathfrak{g}$.

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Examples

- $w = 1 : \; K(\lambda, 1, \mu) \cong V(\lambda + \mu)$.
- $w = w_0$ the longest element (if $\mathfrak{g}$ is finite dimensional) : $K(\lambda, w_0, \mu) = V(\lambda) \otimes V(\mu)$.

Filtration of $V(\lambda) \otimes V(\mu)$ by Kostant modules

Observation 1. Let W_λ and W_μ be the stabilizers of dominant integral weights λ and μ respectively. Then $K(\lambda, w_1, \mu) = K(\lambda, w_2, \mu)$ if $W_\lambda \; w_1 \; W_\mu = W_\lambda \; w_2 \; W_\mu$ for $w_1, w_2 \in W$.

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Observation 2. Let $w_1, w_2 \in W$, then $K(\lambda, w_1, \mu) \subseteq K(\lambda, w_2, \mu)$ if $W_\lambda \ w_1 \ W_\mu \leq W_\lambda \ w_2 \ W_\mu$ in the Bruhat order on $W_\lambda \backslash W / W_\mu$.

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Thus we see that Kostant modules form an increasing filtration of $V(\lambda) \otimes V(\mu)$ by $U\mathfrak{g}$-submodules, indexed by the double coset space $W_\lambda \backslash W / W_\mu$ thought of as a poset under the Bruhat order.

Some basic notions and results due to Littelmann

Let $\mathfrak{B}_\lambda$ be the set of Lakshmibai-Seshadri(L-S) paths of shape λ. Recall that a path $\pi \in \mathfrak{B}_\lambda$ consists of a sequence $\tau_1 > \tau_2 > ... > \tau_r$ of elements of W/W_λ and a sequence of rational numbers $0 = a_0 < a_1 < ... < a_r = 1$. We call τ_1 the initial direction and τ_r the final direction of π.

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Root operators

For every simple root α, Littelmann associated two operators e_α and f_α on the set of paths.

Let $\mathfrak{B}_\lambda * \mathfrak{B}_\mu := \{\pi * \pi' | \pi \in \mathfrak{B}_\lambda, \pi' \in \mathfrak{B}_\mu\}$, where $*$ denotes concatenation, and $\mu \in \Lambda^+$.

Kostant Set

Given a path $\pi * \pi' \in \mathfrak{B}_\lambda * \mathfrak{B}_\mu$, we associate a Weyl group element $\mathfrak{m}(\pi * \pi')$ by:

$$\mathfrak{m}(\pi * \pi') := \textbf{min } W_\lambda \ I(\tau^{-1}) \ \sigma \ W_\mu$$

- $\textbf{min } W_\lambda \ I(\tau^{-1}) \ \sigma \ W_\mu$: unique minimal element of this set (exists).
- τ : lift in W of the final direction of π.
- σ : lift in W of the initial direction of π'.
- $I(\tau^{-1})$: Bruhat interval $\{w \in W | w \leq \tau^{-1}\}$.

Kostant Set

Kostant Set in $\mathfrak{B}_\lambda * \mathfrak{B}_\mu$

Given an element ϕ of the double coset space $W_\lambda \backslash W / W_\mu$, define the corresponding **Kostant set** by:

$$(\mathfrak{B}_\lambda * \mathfrak{B}_\mu)_\phi := \{ \pi * \pi' \in \mathfrak{B}_\lambda * \mathfrak{B}_\mu \mid \mathfrak{m}(\pi * \pi') \leq \widetilde{\phi} \}$$

where $\widetilde{\phi}$ is any lift of ϕ (the choice of lift doesn't matter).

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- $(\mathfrak{B}_\lambda * \mathfrak{B}_\mu)_\phi \subseteq (\mathfrak{B}_\lambda * \mathfrak{B}_\mu)_{\phi'}$ if $\phi \leq \phi'$.
- Kostant sets form an increasing filtration of $\mathfrak{B}_\lambda * \mathfrak{B}_\mu$ indexed by the Bruhat poset $W_\lambda \backslash W / W_\mu$.

Stability of Kostant set under root operators

Lemma [__ , Raghavan,Viswanath, 2018]

Let $\pi * \pi'$ and $\sigma * \sigma'$ be paths in $\mathfrak{B}_\lambda * \mathfrak{B}_\mu$ such that $\sigma * \sigma'$ equals either $e_\alpha(\pi * \pi')$ or $f_\alpha(\pi * \pi')$ for some simple root α. Then $\mathfrak{m}(\pi * \pi') = \mathfrak{m}(\sigma * \sigma')$.

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Equivalence relation on $\mathfrak{B}_\lambda * \mathfrak{B}_\mu$

Let $\pi * \pi'$ and $\sigma * \sigma'$ be the paths in $\mathfrak{B}_\lambda * \mathfrak{B}_\mu$, let us say $\pi * \pi'$ related to $\sigma * \sigma'$, if $\pi * \pi'$ equals either $e_\alpha(\sigma * \sigma')$ or $f_\alpha(\sigma * \sigma')$. This relation is symmetric since $\pi * \pi' = e_\alpha(\sigma * \sigma')$ if and only if $f_\alpha(\pi * \pi') = \sigma * \sigma'$. Denote by $\sim$ the reflexive and transitive closure of this relation (as α varies over all simple roots). Thus a Kostant set is a union of equivalence classes of $\sim$.

Path model for the Kostant module

Given $\phi \in W_\lambda \backslash W / W_\mu$, consider the **Kostant set**

$$(\mathfrak{B}_\lambda * \mathfrak{B}_\mu)_\phi := \{\pi * \pi' \in \mathfrak{B}_\lambda * \mathfrak{B}_\mu | \ \mathfrak{m}(\pi * \pi') \leq \widetilde{\phi}\}$$

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Theorem [__ , Raghavan,Viswanath, 2018]

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra. $(\mathfrak{B}_\lambda * \mathfrak{B}_\mu)_\phi$ is a path model for the Kostant module $K(\lambda, \phi, \mu)$, i.e.,

$$\textit{char } K(\lambda, \phi, \mu) = \textit{char } (\mathfrak{B}_\lambda * \mathfrak{B}_\mu)_\phi := \sum_{\pi \in (\mathfrak{B}_\lambda * \mathfrak{B}_\mu)_\phi} e^{\pi(1)}$$

Decomposition rule for Kostant modules

Theorem [__ , Raghavan, Viswanath, 2018]

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra. Let λ, μ be dominant integral weights and w an element of Weyl group. The decomposition of the Kostant module $K(\lambda, w, \mu)$ as a direct sum of irreducible $\mathfrak{g}$-modules is given by

$$K(\lambda, w, \mu) \cong \bigoplus_{\pi \in \mathfrak{B}_\mu^\lambda(w)} V(\lambda + \pi(1))$$

where $\mathfrak{B}_\mu^\lambda := \{\pi \in \mathfrak{B}_\mu | \ \lambda + \pi(t) \ \textit{dominant for all} \ t \in [0,1]\}$ and $\mathfrak{B}_\mu^\lambda(w) := \{\pi \in \mathfrak{B}_\mu^\lambda | \ \textit{initial direction of} \ \pi \ \textit{is} \ \leq wW_\mu\}$.

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Note: Proof uses Kumar's [Inv. Math, 1988] character formula for the Kostant module and follows Littelmann's [Inv. Math, 1994] proof of the decomposition rule for $V(\lambda) \otimes V(\mu)$.

PRV and refinements

Let $\nu = \overline{\lambda + \sigma\mu}$ (dominant conjugate) for some $\sigma \in W$.

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Kostant's refinement [Kumar 1988]

$V(\nu)$ occurs with multiplicity ≥ 1 in $K(\lambda, w, \mu) \subseteq V(\lambda) \otimes V(\mu)$ for all $W_\lambda\ w\ W_\mu \geq W_\lambda\ \sigma\ W_\mu$ (exactly 1 for $\sigma = w$).

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Verma's refinement [Kumar 1989]

In $V(\lambda) \otimes V(\mu)$, $V(\nu)$ occurs with multiplicity $\geq$ # double cosets $W_\lambda \; \tau \; W_\mu$ such that $\nu = \overline{\lambda + \tau\mu}$.

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Kostant-Verma simultaneous refinement [_ , Raghavan,Viswanath, 2018]

In $K(\lambda, w, \mu)$, $V(\nu)$ occurs with multiplicity $\geq$ # double cosets $W_\lambda \; \tau \; W_\mu$ such that $\nu = \overline{\lambda + \tau\mu}$ and $W_\lambda \; \tau \; W_\mu \leq W_\lambda \; w \; W_\mu$ ($w = w_0$ gives Verma's refinement).

Another consequence

Proposition [Kumar 1988]

Suppose $\lambda, \mu \in \Lambda^+$ are both regular. Fix $w \in W$. Then the $\mathfrak{g}$-module $V(\overline{\lambda + w\mu})$ does not occur in $K(\lambda, v, \mu)$, for any $v < w$.

Proposition [__ , Raghavan,Viswanath, 2018]

Let $\lambda, \mu \in \Lambda^+$ and fix $W_\lambda \ w \ W_\mu \in W_\lambda \backslash W / W_\mu$. Then the $\mathfrak{g}$-module $V(\overline{\lambda + w\mu})$ does not occur in $K(\lambda, v, \mu)$, for any $W_\lambda \ v \ W_\mu < \ W_\lambda \ w \ W_\mu$.

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Thank You