Block Decomposition of a Class of Integrable Representations of Toroidal Lie Algebras

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Interactions of quantum affine algebras with cluster algebras, current algebras and categorification

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Conference celebrating 60th birthday of Vyjayanthi Chari

Notations

Let

- $\mathfrak{g}$ a complex finite-dimensional simple Lie algebra ;
- $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$.
- $\{\alpha_i : 1 \leq i \leq n\}$:= simple roots of $\mathfrak{g}$,
- $\{\omega_1, \cdots, \omega_n\}$:= fundamental weights of $\mathfrak{g}$,
- $Q_{fin} = \sum_{i=1}^{n} \mathbb{Z}\alpha_i$ the root lattice, $P_{fin} = \sum_{i=1}^{n} \mathbb{Z}\omega_i$, weight lattice and $P^+_{fin} = \sum_{i=1}^{n} \mathbb{Z}_+\omega_i$ dominant integral weights of $\mathfrak{g}$
- θ the highest root of $\mathfrak{g}$ and $\theta^\vee$ the corresponding co-root;

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- $\mathfrak{g}_{aff}$ affine Kac-Moody algebra associated with $\mathfrak{g}$
- $\mathfrak{h}_{aff}$ a Cartan subalgebra of $\mathfrak{g}_{aff}$
- $(.|.)$ the Killing form on $\mathfrak{g}$;
- $\{\alpha_i : 1 \leq 0 \leq n\}$:= simple roots of $\mathfrak{g}_{aff}$
- $\{\Lambda_1, \cdots, \Lambda_n, \Lambda_0\}$:= fundamental weights of $\mathfrak{g}_{aff}$
- $Q_{aff} = \sum_{i=0}^{n} \mathbb{Z}\alpha_i$ the root lattice $P_{aff} = \sum_{i=0}^{n} \mathbb{Z}\Lambda_i$, weight lattice and $P^+_{aff} = \sum_{i=0}^{n} \mathbb{Z}_+\Lambda_i$ dominant integral weights of $\mathfrak{g}_{aff}$

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Toroidal Lie Algebra

Definition

A k-toroidal Lie algebra associated with $\mathfrak{g}$ is a Lie algebra with underlying vector space

$$\mathcal{T}_k(\mathfrak{g}) := \mathfrak{g} \otimes \mathbb{C}[t_1^{\pm 1}, \cdots, t_k^{\pm k}] \oplus D_k \oplus \mathcal{Z},$$

where,

- D_k is the space spanned by k derivations $d_1, \cdots, d_k$,
- $\mathcal{Z}$ is an infinite-dimensional space spanned by $\mathbb{Z}^k$-graded central elements $\{t^{\mathbf{m}} c_i, \mathbf{m} \in \mathbb{Z}^k, 1 \leq i \leq k\}$, together with the relation $\sum_{i=1}^k r_i t^{\mathbf{r}} c_i = 0$; and
- Lie bracket :

$$[x \otimes t^{\mathbf{m}}, y \otimes t^{\mathbf{s}}] = [x, y] \otimes t^{\mathbf{m}+\mathbf{s}} + \sum_{i=1}^{k} m_i t^{\mathbf{m}+\mathbf{s}} c_i \ (x|y),$$

$$d_i(x \otimes t^{\mathbf{m}}) = m_i x \otimes t^{\mathbf{m}}, \qquad \forall\, x \in \mathfrak{g}.$$

Let $\mathfrak{h}_{tor} := \mathfrak{h} \oplus D_k \oplus \mathbb{C}c_1 \oplus \cdots \oplus \mathbb{C}c_k$ where $c_1, \cdots, c_k$ are the zero graded central elements in $\mathcal{T}_k(\mathfrak{g})$.

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Integrable Representation of Lie algebra

Definition

A $\mathcal{T}_k(\mathfrak{g})$-module V is said integrable if

- $V = \bigoplus_{\mu \in \mathfrak{h}^*_{tor}} V_\mu$, where $V_\mu = \{v \in V : h.v = \mu(h)v,$ for all $h \in \mathfrak{h}\}$.
- the root vectors corresponding to the real roots of $\mathcal{T}_k(\mathfrak{g})$ act nilpotently on every non-zero vector of V.

For an integer k, let $\mathbb{I}^k_{fin}$ be the category of integral $\mathcal{T}_k(\mathfrak{g})$-modules with finite-dimensional weight spaces.

Irreducible objects of $\mathbb{I}^1_{fin}$

For $k = 1$,
$\mathcal{T}_k(\mathfrak{g}) = \mathfrak{g}_{aff}$ (untwisted affine Kac-Moody algebra associated with $\mathfrak{g}$):

V. Chari, A. Pressley

- The simple objects of $\mathbb{I}^1_{fin}$ on which the center acts trivially are of the form

$$V(\vec{\lambda}, \vec{b}, s) := V_{b_1}(\lambda_1) \otimes \cdots \otimes V_{b_r}(\lambda_r) \otimes t^s \mathbb{C}[t_1^{\pm m}]$$

  where, $\vec{\lambda} = (\lambda_1, \cdots, \lambda_r) \in (P^+_{fin})^r$, $\vec{b} = (b_1, \cdots, b_r) \in (\mathbb{C}^\times)^r$ for $r \in \mathbb{Z}_+$, $m \in \mathbb{Z}_+$ and $0 \leq s \leq m - 1$,
  and
- the simple objects of $\mathbb{I}^1_{fin}$ on which the center acts non-trivially are :
  - standard modules of the form $X(\Lambda)$, with $\Lambda \in P^+_{aff}$ or
  - restricted duals of standard modules.

Here, for $\lambda \in P^+$, $b \in \mathbb{C}^\times$, $V_b(\lambda)$ is the evaluation module for the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t_1^{\pm 1}]$, with underlying vector space $V(\lambda)$, and $\mathfrak{g} \otimes \mathbb{C}[t_1^{\pm 1}]$ action:

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Block Decomposition of the category $\mathbb{I}_{fin}^{1}[0]$

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- An indecomposable object in $\mathbb{I}^1_{fin}[0]$ has finitely many simple constituents which are non-trivial as modules over $\mathfrak{g} \otimes \mathbb{C}[t_1, t_1^{-1}]$.
- Given two irreducible $\mathfrak{g}_{aff}$-modules,

$$V(\vec{\lambda}, \vec{b}, s) \quad \text{and} \quad V(\vec{\mu}, \vec{a}, p),$$

  with $\vec{\lambda}, \vec{\mu} \in (P^+_{fin})^r$ and $\vec{b}, \vec{a} \in (\mathbb{C}^\times)^r$, there exists a sequence of indecomposable $\mathfrak{g}_{aff}$-modules $V_1, V_2, \cdots, V_r$ in $\mathbb{I}^1_{fin}[0]$ such that

$$Hom(V_i, V_{i+1}) \neq 0, \quad \text{or} \quad Hom(V_{i+1}, V_i) \neq 0,$$

  if and only if

  - $\vec{b} = s.\vec{a}$ for some $s \in \mathbb{C}^*$ and
  - $\vec{\lambda} - \vec{\mu} \in (Q_{fin})^r$.

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- An indecomposable object in $\mathbb{I}^1_{fin}[0]$ has finitely many simple constituents which are non-trivial as modules over $\mathfrak{g} \otimes \mathbb{C}[t_1, t_1^{-1}]$.
- The category $\mathbb{I}^1_{fin}[0]$ decomposes into blocks which are parametrized by orbits for the natural action of the group $\mathbb{C}^\times$ on

$$\Xi := \{f : \mathbb{C}^\times \to P_{fin}/Q_{fin} : f(a) = 0 \text{ for all but finitely many } a \in \mathbb{C}^\times\}.$$

Blocks in $\mathbb{I}^1_{fin}[0]$ $\longleftrightarrow$ Equivalence classes of indecomposable objects in $\mathbb{I}^1_{fin}[0]$ with respect to the equivalence relation $\sim$ defined as follows.

$X \sim Y$ if there exists a sequence of indecomposable modules $X = X_1, \cdots, X_k = Y$ in $\mathbb{I}^1_{fin}[0]$ such that for $1 \leq i \leq k-1$,

$$Hom(X_i, X_{i+1}) \neq 0, \quad \text{or} \quad Hom(X_{i+1}, X_i) \neq 0$$

Irreducible Modules in $\mathbb{I}^k_{fin}$ for $k > 1$

S. E Rao, 2004

For $k \geq 2$,

- the simple objects in $\mathbb{I}^k_{fin}$ on which the central elements act trivially are of the form,

$$V(\vec{\lambda}, \vec{\mathbf{a}}, \mathbf{s}) := V_{\mathbf{a}_1}(\lambda_1) \otimes \cdots \otimes V_{\mathbf{a}_r}(\lambda_r) \otimes t^{\mathbf{s}}\mathbb{C}[t^{G(\vec{\lambda}, \vec{\mathbf{a}})}]$$

where, $\vec{\lambda} = (\lambda_1, \cdots, \lambda_r) \in (P^+_{fin})^r$, $\vec{\mathbf{a}} = (\mathbf{a}_1, \cdots, \mathbf{a}_r) \in ((\mathbb{C}^\times)^k)^r$ for $r \in \mathbb{Z}_+$, $\mathbf{G}(\vec{\lambda}, \vec{\mathbf{a}})$ is a subgroup of $\mathbb{Z}^k$ of rank k and $t^{\mathbf{s}} \in \mathbb{C}[t_1^{\pm 1}, \cdots, t_k^{\pm 1}]$.

- $V_{\mathbf{a}}(\lambda)$ is the evaluation module for $\mathfrak{g} \otimes \mathbb{C}[t_1^{\pm 1}, \cdots, t_k^{\pm 1}]$, with underlying vector space $V(\lambda)$, with $\lambda \in P^+_{fin}$, $\mathbf{a} \in (\mathbb{C}^\times)^k$.

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- Upto an isomorphism, the irreducible objects of $\mathbb{I}^k_{fin}$ on which the center acts non-trivially, are of the form

$$X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0) := X_{\mathfrak{m}_1}(\Lambda_1) \otimes \cdots X_{\mathfrak{m}_r}(\Lambda_r) \otimes t^{\mathbf{r}_0}\mathbb{C}[t^{G(\vec{\Lambda}, \vec{\mathfrak{m}})}],$$

  where $\underline{\Lambda} = (\Lambda_1, \cdots, \Lambda_r) \in P^+_{aff}$, $\vec{\mathfrak{m}} = (\mathfrak{m}_1, \cdots, \mathfrak{m}_r) \in (\max \mathbb{C}[t_2^{\pm 1}, \cdots, t_k^{\pm 1}])^r$, $G(\underline{\Lambda}, \vec{\mathfrak{m}})$ is a subgroup of $\mathbb{Z}^{k-1}$ of rank $k-1$ and $t^{\mathbf{r}_0} \in \mathbb{C}[t_2^{\pm 1}, \cdots, t_k^{\pm 1}]$.

- $X_{\mathfrak{m}}(\Lambda)$ is the evaluation module for the algebra $\mathfrak{g}'_{aff} \otimes \mathbb{C}[t_2^{\pm 1}, \cdots, t_k^{\pm 1}]$, with underlying vector space $X(\Lambda)$, with $\Lambda \in P^+_{aff}$, $\mathfrak{m} \in \max \mathbb{C}[t_2^{\pm 1}, \cdots, t_k^{\pm 1}]$.

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$$Irr(\mathbb{J}^{+}_{fin,k}) := \{ X(\overrightarrow{\Lambda}, \overrightarrow{\mathfrak{m}}, \mathbf{r}_0) : \overrightarrow{\Lambda} \in (P^{+}_{aff})^{s}, \overrightarrow{\mathfrak{m}} \in (\max \mathbb{C}[t_2^{\pm 1}, \cdots, t_k^{\pm 1}])^{s}, \mathbf{r}_0 \in \mathbb{Z}^{k-1} \}$$

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For $X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0) \in Irr(\mathbb{J}^+_{fin,k})$ and $\mathfrak{m} \in \max \mathbb{C}[t_2^{\pm 1}, \cdots, t_k^{\pm 1}]$

let,

- $supp(X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0)) = \{\mathfrak{m}_1, \cdots, \mathfrak{m}_r\}$
- $wt_{X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0)}(\mathfrak{m}) = \begin{cases} \Lambda & \text{if } X_{\mathfrak{m}}(\Lambda) \text{ is a tensor component of } X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0) \\ 0 & \text{otherwise} \end{cases}$

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Recall $\Lambda \in P_{aff}$ is of the form

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Type I and Type II Irreducibles in $\mathbb{J}^{+}_{fin}$

$X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0) \in Irr(\mathbb{J}^{+}_{fin,k})$ is said to be of

- Type I, if $wt_{X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0)}(\mathfrak{m})(c_1) < \theta(\theta^\vee)$ for all $\mathfrak{m} \in supp(X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0))$;
- Type II, otherwise.

Results

Using results by Adamovich on the irreducibility of the tensor product of a highest weight irreducible integrable $\mathfrak{g}_{aff}$-module and the loop modules for $\mathfrak{g}_{aff}$, we prove:

Result

Let $X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0)$ be an irreducible $\mathcal{T}_k(\mathfrak{g})$ of type I. Then

- $(\mathfrak{g} \otimes \mathbb{C}[t_1^{\pm 1}]) \otimes X_{\mathfrak{m}}(\Lambda)$ is an irreducible $\mathfrak{g}_{aff}$-module for all $\mathfrak{m} \in supp(X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0))$.
- The local Weyl module corresponding to $X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0)$ is irreducible.
- $Ext^1(X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0), X(\vec{\Lambda}', \vec{\mathfrak{m}}', \mathbf{s}_0)) = 0$, if $X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0) \not\cong_{\mathcal{T}_k(\mathfrak{g})} X(\vec{\Lambda}', \vec{\mathfrak{m}}', \mathbf{s}_0)$

Theorem 1.

If V is an indecomposable $\mathcal{T}_k(\mathfrak{g})$-module in $\mathbb{J}^+_{fin,k}$, such that a type I irreducible $\mathcal{T}_k(\mathfrak{g})$-module, $X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0)$, is an irreducible constituent of V, then every irreducible constituent of V is isomorphic to $X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0)$.

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Result

Let $X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0)$ be an irreducible $\mathcal{T}_k(\mathfrak{g})$ of type I. Then

- $(\mathfrak{g} \otimes \mathbb{C}[t_1^{\pm 1}]) \otimes X_{\mathfrak{m}}(\Lambda)$ is an irreducible $\mathfrak{g}_{aff}$-module for all $\mathfrak{m} \in supp(X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0))$.
- The local Weyl module corresponding to $X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0)$ is irreducible.
- $Ext^1(X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0), X(\vec{\Lambda'}, \vec{\mathfrak{m}'}, \mathbf{s}_0)) = 0$, if $X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0) \ncong_{\mathcal{T}_k(\mathfrak{g})} X(\vec{\Lambda'}, \vec{\mathfrak{m}'}, \mathbf{s}_0)$

Theorem 1.

If V is an indecomposable $\mathcal{T}_k(\mathfrak{g})$-module in $\mathbb{J}^+_{fin,k}$, such that a type I irreducible $\mathcal{T}_k(\mathfrak{g})$-module, $X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0)$, is an irreducible constituent of V, then every irreducible constituent of V is isomorphic to $X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0)$.

Theorem 2.

Let $\mathcal{T}_k(\mathfrak{g})$ be a toroidal Lie algebra where the underlying finite-dimensional Lie algebra is of type A_n, D_n, E_7, E_8 or F_4.

Given two irreducible $\mathcal{T}_k(\mathfrak{g})$-modules of type II,

$$X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0), \quad \text{and} \quad X(\vec{\Lambda}', \vec{\mathfrak{m}}', \mathbf{s}_0),$$

such that

- $\vec{\mathfrak{m}}' = \mathbf{c}.\vec{\mathfrak{m}}$ for some $\mathbf{c} \in (C^*)^{k-1}$,
- $(\vec{\Lambda} - \vec{\Lambda}')(c_1) = 0$, and $(\vec{\Lambda} - \vec{\Lambda}')|_{\mathfrak{h}} \in (Q_{fin})^r$,

there exists a sequence $X_1, X_2, \cdots, X_r$ of irreducible $\mathcal{T}_k(\mathfrak{g})$-modules of type II, with $X_r = X(\vec{\Lambda}', \vec{\mathfrak{m}}', \mathbf{s}_0)$ such that upto tensoring by one-dimensional modules $X_1 \cong_{\mathcal{T}_k(\mathfrak{g})} X(\vec{\Lambda}, \vec{\mathfrak{m}}, \mathbf{r}_0)$ and for each $1 \leq j \leq r-1$,

$$Ext^i(X_j, X_{j+1}) \neq 0 \quad \text{or} \quad Ext^i(X_{j+1}, X_j) \neq 0, \qquad \text{for } i = 0 \text{ or } 1.$$

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Thank you !