On the affine VW supercategory

Mee Seong Im
West Point, NY

Interactions of quantum affine algebras with cluster algebras, current algebras and categorification
A conference celebrating the 60th birthday of Vyjayanthi Chari
Catholic University of America, Washington, D.C.

May 28, 2018

Joint with
Martina Balagovic, Zajj Daugherty, Inna Entova-Aizenbud,
Iva Halacheva, Johanna Hennig, Gail Letzter,
Emily Norton, Vera Serganova, and Catharina Stroppel.

Background: vector superspaces. Work over $\mathbb{C}$.

A $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V = V_{\overline{0}} \oplus V_{\overline{1}}$ is a *vector superspace*.

The superdimension of V is
$\dim(V) := (\dim V_{\overline{0}} | \dim V_{\overline{1}}) = \dim V_{\overline{0}} - \dim V_{\overline{1}}$.

Given a homogeneous element $v \in V$, the *parity* (or the *degree*) of v is $\overline{v} \in \{\overline{0}, \overline{1}\}$.
The parity switching functor π sends $V_{\overline{0}} \mapsto V_{\overline{1}}$ and $V_{\overline{1}} \mapsto V_{\overline{0}}$.

Let $m = \dim V_{\overline{0}}$ and $n = \dim V_{\overline{1}}$. The Lie superalgebra is $\mathfrak{gl}(m|n) := \mathrm{End}_{\mathbb{C}}(V)$.

That is, given a homogeneous ordered basis for V:

$$V = \underbrace{\mathbb{C}\{v_1, \ldots, v_m\}}_{V_{\overline{0}}} \oplus \underbrace{\mathbb{C}\{v_{1'}, \ldots, v_{n'}\}}_{V_{\overline{1}}},$$

Matrix representation for $\mathfrak{gl}(m|n)$.

the *Lie superalgebra* is the endomorphism algebra

$$\mathfrak{gl}(m|n) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A \in M_{m,m}, B, C^t \in M_{m,n}, D \in M_{n,n} \right\},$$

where $M_{i,j} := M_{i,j}(\mathbb{C})$. Since $\mathfrak{gl}(m|n) = \mathfrak{gl}(m|n)_{\overline{0}} \oplus \mathfrak{gl}(m|n)_{\overline{1}}$,

$$\mathfrak{gl}(m|n)_{\overline{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\} \text{ and } \mathfrak{gl}(m|n)_{\overline{1}} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\}.$$

We say V is the *natural representation* of $\mathfrak{gl}(m|n)$.

The grading on $\mathfrak{gl}(m|n)$ is induced by V, with *Lie superbracket* (supercommutator) $[x, y] = xy - (-1)^{\overline{xy}} yx$ for x, y homogeneous.

Periplectic Lie superalgebras $\mathfrak{p}(n)$.

Let $m = n$. Then

$$V = \mathbb{C}^{2n} = \underbrace{\mathbb{C}\{v_1, \ldots, v_n\}}_{V_{\overline{0}}} \oplus \underbrace{\mathbb{C}\{v_{1'}, \ldots, v_{n'}\}}_{V_{\overline{1}}}.$$

Define $\beta : V \otimes V \to \mathbb{C}$ as a symmetric, odd, nondegenerate bilinear form satisfying:

$$\beta(v, w) = \beta(w, v), \qquad \beta(v, w) = 0 \quad \text{if } \overline{v} = \overline{w}.$$

We define *periplectic (strange) Lie superalgebras* as:

$$\mathfrak{p}(n) := \{x \in \mathrm{End}_{\mathbb{C}}(V) : \beta(xv, w) + (-1)^{\overline{xv}}\beta(v, xw) = 0\}.$$

In terms of above basis,

$$\mathfrak{p}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{gl}(n|n) : B = B^t, C = -C^t \right\}.$$

Symmetric monoidal structure.

Consider the category $\mathcal{C}$ of representations of $\mathfrak{p}(n)$ with

$$\mathrm{Hom}_{\mathfrak{p}(n)}(V, V') := \{f : V \to V' : f \text{ homogeneous}, \mathbb{C} - \text{linear}, \\ f(x.v) = (-1)^{\overline{xf}} x.f(v), v \in V, x \in \mathfrak{p}(n)\}.$$

Then $U(\mathfrak{p}(n))$ of $\mathfrak{p}(n)$ is a Hopf superalgebra:

- (coproduct) $\Delta(x) = x \otimes 1 + 1 \otimes x$,
- (counit) $\epsilon(x) = 0$,
- (antipode) S(x) = -x.

So the category of representations of $\mathfrak{p}(n)$ is monoidal.
For $x \otimes y \in U(\mathfrak{p}(n)) \otimes U(\mathfrak{p}(n))$ on $v \otimes w$,

$$(x \otimes y).(v \otimes w) = (-1)^{\overline{yv}} xv \otimes yw.$$

Symmetric monoidal structure.

For $x, y, a, b \in U(\mathfrak{p}(n))$,

$$(x \otimes y) \circ (a \otimes b) := (-1)^{\overline{ya}}(x \circ a) \otimes (y \circ b),$$

and for two representations V and V', the *super swap*

$$\sigma : V \otimes V' \longrightarrow V' \otimes V, \quad \sigma(v \otimes w) = (-1)^{\overline{vw}} w \otimes v$$

is a map of $\mathfrak{p}(n)$-representations satisfying $\sigma^* = -\sigma$.
Thus $\mathcal{C}$ is a symmetric monoidal category.
Furthermore, β induces a representation V and its dual V^* via

$$V \to V^*, \quad v \mapsto \beta(v, -),$$

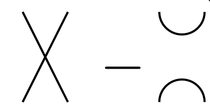
identifying $V_{\overline{1}}$ with $V_{\overline{0}}^*$ and $V_{\overline{0}}$ with $V_{\overline{1}}^*$. This induces the dual map

$$\beta^* : \mathbb{C} \cong \mathbb{C}^* \longrightarrow (V \otimes V)^* \cong V \otimes V, \quad \beta^*(1) = \sum_i -v_i \otimes v_{i'} + v_{i'} \otimes v_i,$$

where $\overline{\beta} = \overline{\beta^*} = 1$.

Quadratic (fake) Casimir and Jucys-Murphy elements: y_ℓ's.

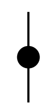
Furthermore, we define

$$\Omega = 2\sum_{x \in \mathcal{X}} x \otimes x^* \in \mathfrak{p}(n) \otimes \mathfrak{gl}(n|n) \quad \Big(2\Omega = $$  $$\Big),$$

where $\mathcal{X}$ is a basis of $\mathfrak{p}(n)$ and x^* is a dual basis element of $\mathfrak{p}(n)$, and $\mathfrak{p}(n)^\perp$ is taken with respect to the supertrace:

$$\mathrm{str}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{tr}(A) - \mathrm{tr}(D).$$

The actions of Ω and $\mathfrak{p}(n)$ commute on $M \otimes V$, so $\Omega \in \mathrm{End}_{\mathfrak{p}(n)}(M \otimes V)$. We define

$$Y_\ell : M \otimes V^{\otimes a} \longrightarrow M \otimes V^{\otimes a} \text{ as } Y_\ell = \sum_{i=0}^{\ell-1} \Omega_{i,\ell} = $$  ,

where $\Omega_{i,\ell}$ acts on the i-th and ℓ-th factor, and identity otherwise, where the 0-th factor is the module M.

Classical Schur-Weyl duality.

Let W be an n-dimensional complex vector space. Consider $W^{\otimes a}$. Then the symmetric group S_a acts on $W^{\otimes a}$ by permuting the factors: for $s_i = (i \;\; i+1) \in S_a$,

$$s_i.(w_1 \otimes \cdots \otimes w_a) = w_1 \otimes \cdots \otimes w_{i+1} \otimes w_i \otimes \cdots \otimes w_a.$$

We also have $GL(W)$ acting on $W^{\otimes a}$ via the diagonal action: for $g \in GL(W)$,

$$g.(w_1 \otimes \cdots \otimes w_a) = gw_1 \otimes \cdots \otimes gw_a.$$

Then actions of $GL(W)$ (left natural action) and S_a (right permutation action) commute giving us the following:

Schur-Weyl duality.

Consider the natural representations

$$(\mathbb{C}S_a)^{op} \xrightarrow{\phi} \mathrm{End}_{\mathbb{C}}(W^{\otimes a}) \quad \text{and} \quad GL(W) \xrightarrow{\psi} \mathrm{End}_{\mathbb{C}}(W^{\otimes a}).$$

Then Schur-Weyl duality gives us

1. $\phi(\mathbb{C}S_a) = \mathrm{End}_{GL(W)}(W^{\otimes a})$,
2. if $n \geq a$, then ϕ is injective. So $\operatorname{im}\phi \cong \mathrm{End}_{GL(W)}(W^{\otimes a})$,
3. $\psi(GL(W)) = \mathrm{End}_{\mathbb{C}S_a}(W^{\otimes a})$,
4. there is an irreducible $(GL(W), (\mathbb{C}S_a)^{op})$-bimodule decomposition (see next slide):

Schur-Weyl duality (continued).

$$W^{\otimes a} = \bigoplus_{\substack{\lambda=(\lambda_1,\lambda_2,\dots)\vdash a \\ \ell(\lambda)\leq n}} \Delta_\lambda \otimes S^\lambda,$$

where

- Δ_λ is an irreducible $GL(W)$-module associated to λ,
- S^λ is an irreducible $\mathbb{C}S_a$-module associated to λ, and
- $\ell(\lambda) = \max\{i \in \mathbb{Z} : \lambda_i \neq 0, \lambda = (\lambda_1, \lambda_2, \dots)\}$.

In **higher** Schur-Weyl duality, we construct a result analogous to

$$\mathbb{C}S_a \cong \mathrm{End}_{GL(W)}(W^{\otimes a}),$$

but we use the existence of commuting actions on the tensor product of arbitrary $\mathfrak{gl}_n$-representation M with $W^{\otimes a}$:

$$\mathfrak{gl}_n \circlearrowright M \otimes W^{\otimes a} \circlearrowleft H_a,$$

where H_a is the *degenerate affine Hecke algebra*. The Hecke algebra H_a contains the group algebra $\mathbb{C}S_a$ and the polynomial algebra $\mathbb{C}[y_1, \ldots, y_a]$ as subalgebras.

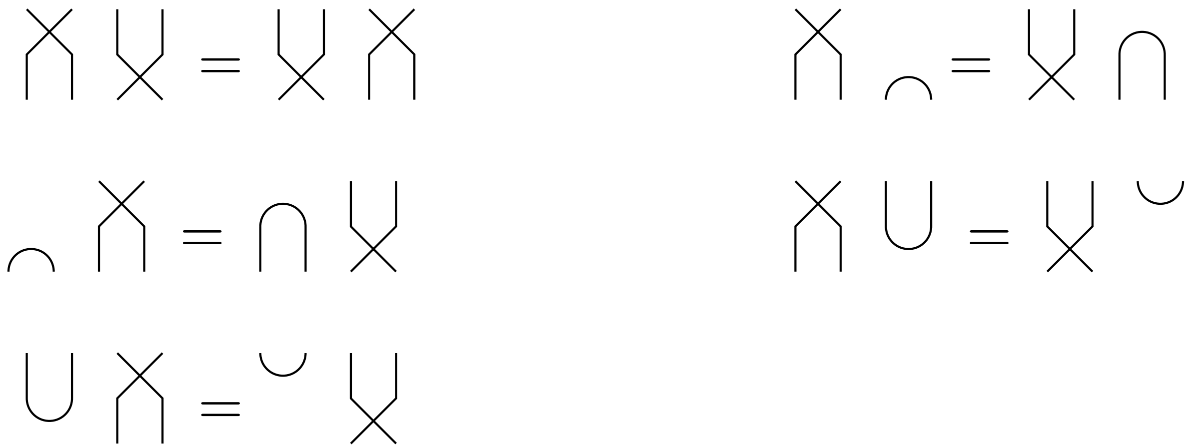
So as a vector space, $H_a \cong \mathbb{C}S_a \otimes \mathbb{C}[y_1, \ldots, y_a]$, and has a basis

$$\mathcal{B} = \{wy_1^{k_1} \cdots y_a^{k_a} : w \in S_a, k_i \in \mathbb{N}_0\}.$$

In this talk, we aim to construct higher Schur-Weyl duality in the context of $\mathfrak{p}(n)$ and affine Brauer algebras, which we will denote by $s\mathbb{W}_a$ (so affine Brauer algebras were constructed from the motivation to formulate higher Schur-Weyl duality for the periplectic Lie superalgebra action, i.e., we need to find another algebra whose action on a representation $M \otimes V^{\otimes a}$ commutes with the action of $\mathfrak{p}(n)$).

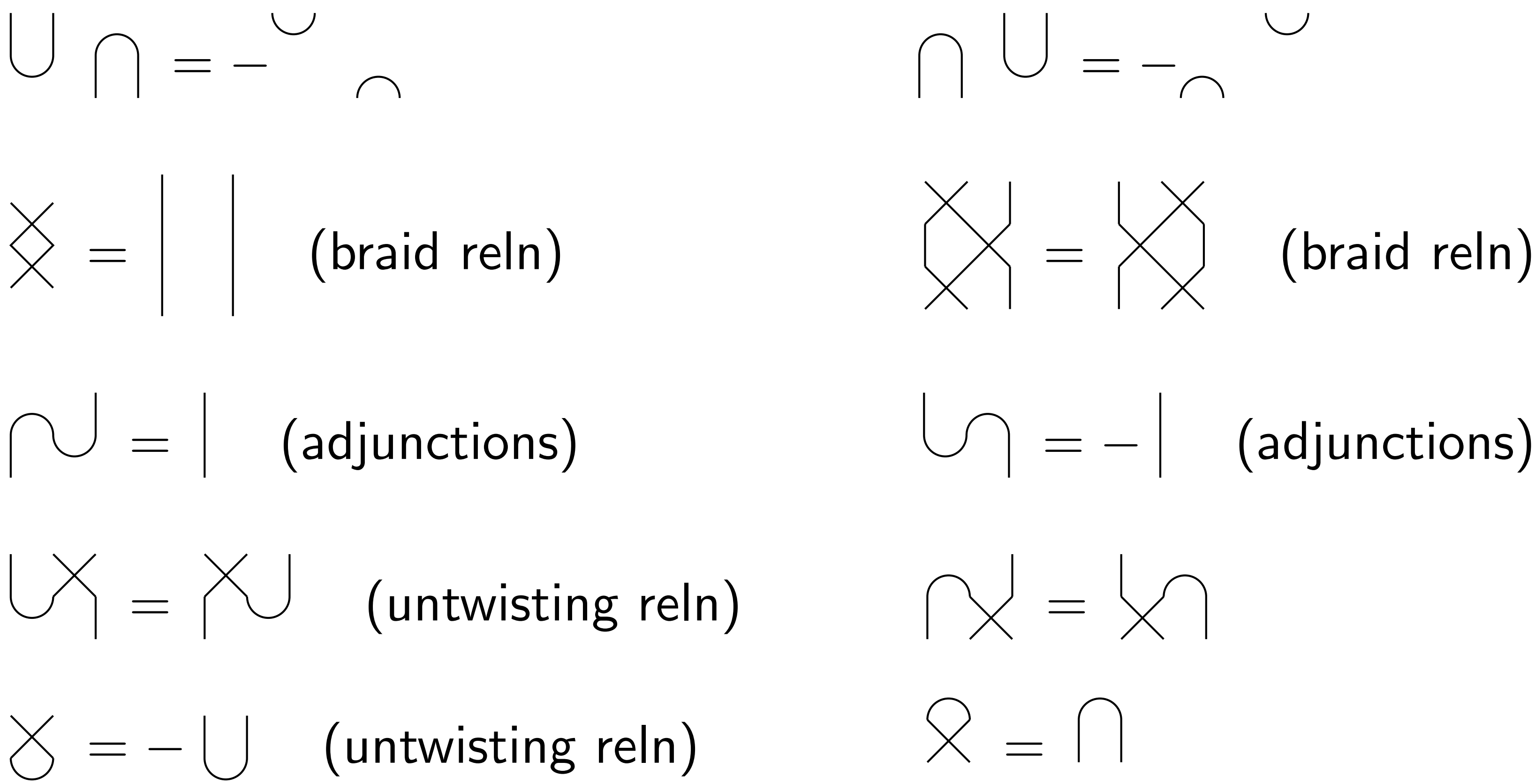
Affine Brauer algebras (generators and local moves).

$s\mathbb{W}_a$ has generators s_i, b_i, b_i^*, y_j, where $i = 1, \ldots, a-1$, $j = 1, \ldots, a$ and relations

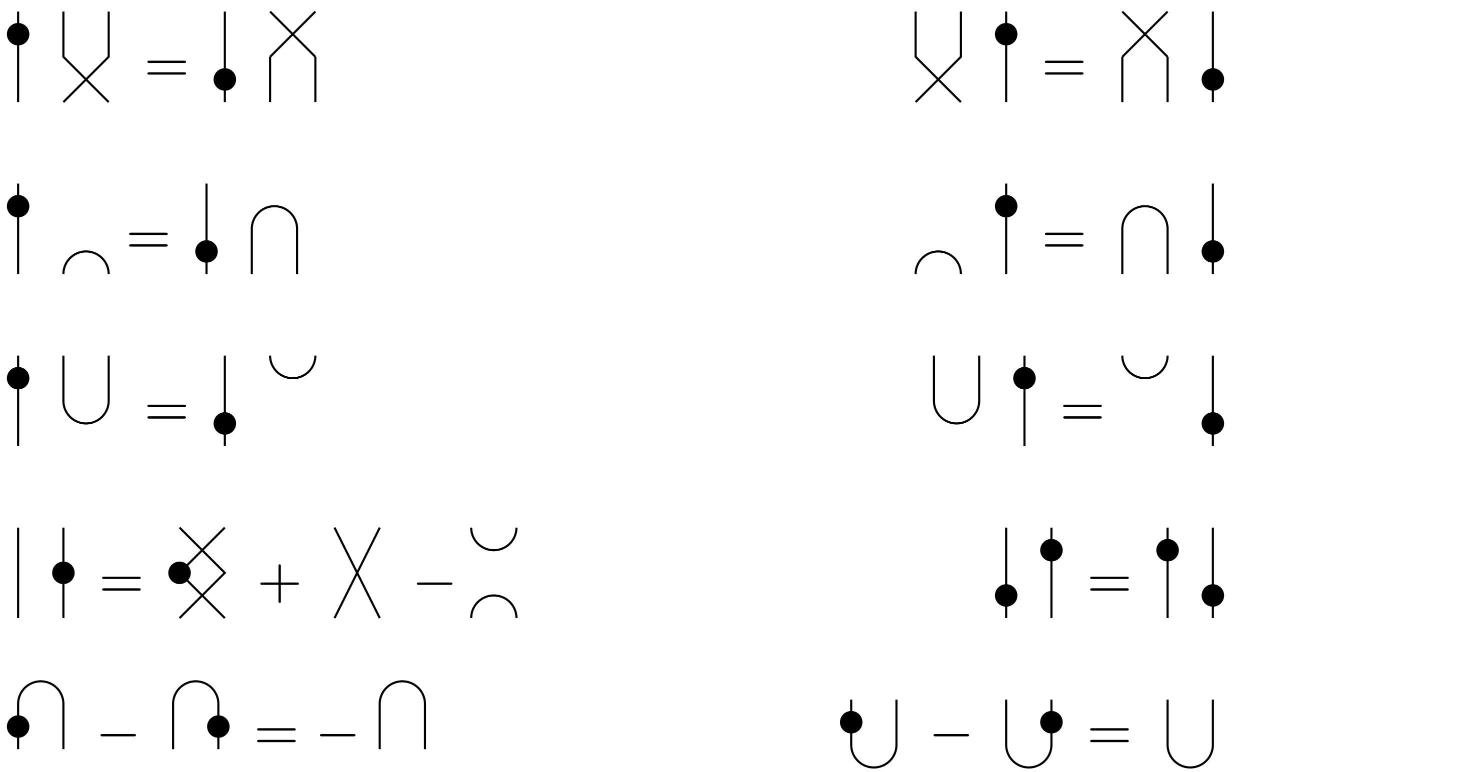


Continued in the next slide.

Affine Brauer algebras (local moves; continued).

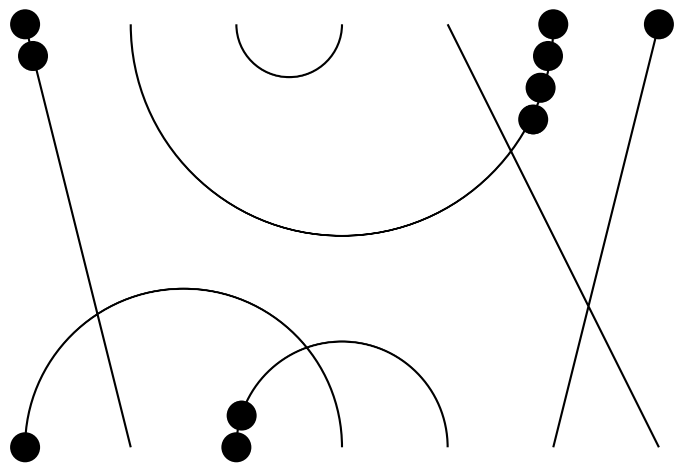


Affine Brauer algebras (local moves; continued).



(Regular) monomials.

An example.



Algebraically, it is written as $y_1^2 y_6^4 y_7 s_5 b_2^* b_2 b_4^* b_4 s_1 s_3 s_6 y_1 y_3^2$.

Our affine VW superalgebra $s\mathbb{W}_a$ is:

- super (signed) version of the degenerate BMW algebra,
- the signed version of the affine VW algebra, and
- an affine version of the Brauer superalgebra.

The center of $s\mathbb{W}_a$.

Theorem

The center $Z(s\mathbb{W}_a)$ consists of all polynomials of the form

$$\prod_{1\leq i<j\leq a}((y_i - y_j)^2 - 1)\widetilde{f} + c,$$

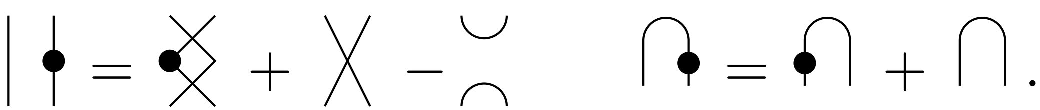
where $\widetilde{f} \in \mathbb{C}[y_1, \ldots, y_a]^{S_a}$ and $c \in \mathbb{C}$.

The deformed squared Vandermonde determinant $\prod_{1\leq i<j\leq a}((y_i - y_j)^2 - 1)$ is symmetric, so

$$\prod_{1\leq i<j\leq a}((y_i - y_j)^2 - 1) \in \mathbb{C}[y_1, \ldots, y_a]^{S_a}.$$

Affine VW supercategory $s\mathbb{W}$ and connections to Brauer supercategory $s\mathcal{B}r$

The affine VW supercategory (or the affine Nazarov-Wenzl supercategory) is the $\mathbb{C}$-linear strict monoidal supercategory generated as a monoidal supercategory by a single object $\bigstar$, morphisms $s = \times : \bigstar \otimes \bigstar \longrightarrow \bigstar \otimes \bigstar$, $\flat = \frown : \bigstar \otimes \bigstar \to \mathbf{1}$, $\flat^* = \smile : \mathbf{1} \to \bigstar \otimes \bigstar$, and an additional morphism $y = \phi : \bigstar \otimes \bigstar \longrightarrow \bigstar \otimes \bigstar$, subject to the braid, snake (adjunction), and untwisting relations, and the dot relations:



Objects in $s\mathbb{W}$ can be identified with natural numbers, identifying $a \in \mathbb{N}_0$ with $\bigstar^{\otimes a}$, $\bigstar^{\otimes 0} = \mathbf{1}$, and the morphisms are linear combinations of dotted diagrams.

$s\mathbb{W}$ and $s\mathcal{B}r$

The category $s\mathbb{W}$ can alternatively be generated by vertically stacking $\flat_i$, $\flat_i^*$, s_i, and $y_i = 1_{i-1} \otimes y \otimes 1_{a-i} \in \mathrm{Hom}_{s\mathbb{W}}(a, a)$.

It is a filtered category, i.e., the hom spaces $\mathrm{Hom}_{s\mathbb{W}}(a, b)$ have a filtration by the span $\mathrm{Hom}_{s\mathbb{W}}(a, b)^{\leq k}$ of all dotted diagrams with at most k dots.

The Brauer supercategory $s\mathcal{B}r$ is the $\mathbb{C}$-linear strict monoidal supercategory generated as a monoidal supercategory by a single object $\bigstar$, and morphisms $s = \times : \bigstar \otimes \bigstar \longrightarrow \bigstar \otimes \bigstar$, $\flat = \frown : \bigstar \otimes \bigstar \to \mathbf{1}$, and $\flat^* = \smile : \mathbf{1} \to \bigstar \otimes \bigstar$, subject to the relations above.

If M is the trivial representation, then actions on $s\mathbb{W}$ factor through $s\mathcal{B}r$.

Thank you.

Questions?