Degeneration of Bethe subalgebras in the Yangian

Aleksei Ilin

National Research University
Higher School of Economics
Faculty of Mathematics
Moscow, Russia

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Yangian for $\mathfrak{gl}_n$

Let $V = \mathbb{C}^n$, $R(u) = 1 - Pu^{-1} \in \operatorname{End}(V \otimes V)[[u^{-1}]]$, where $P(u \otimes v) = v \otimes u$.

Definition

Yangian $Y(\mathfrak{gl}_n)$ *for* $\mathfrak{gl}_n$ is a complex unital associative algebra with countably many generators $t_{ij}^{(1)}, t_{ij}^{(2)}, \dots$ where $1 \leqslant i, j \leqslant n$, and the defining relations

$$R(u-v)T_1(u)T_2(v) = T_1(u)T_2(v)R(u-v).$$

where $T(u) = (t_{ij}(u))_{i,j=1}^n$,

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)}u^{-1} + t_{ij}^{(2)}u^{-2} + \dots \in Y(\mathfrak{gl}_n)[[u^{-1}]].$$

Bethe subalgebras

Let $A_k = \sum_{\sigma \in S_k} (-1)^\sigma \sigma \in \mathbb{C}[S_k]$.

Definition

Consider $C \in \mathfrak{gl}_n$. For any $1 \leqslant k \leqslant n$ introduce the series with coefficients in $Y(gl_n)$ by

$$\tau_k(u, C) = \frac{1}{k!} \operatorname{tr} A_k C_1 \dots C_k T_1(u) \dots T_k(u - k + 1),$$

where we take the trace over all copies of $\mathrm{End}\, \mathbb{C}^{\mathrm{n}}$. We call the subalgebra generated by the coefficients of $\tau_k(u, C)$ *Bethe subalgebra* and denote it by $B(C)$.

Lemma

$$\tau_k(u, C) = \sum_{1 \leqslant a_1 < \dots < a_k \leqslant n} \lambda_{a_1} \dots \lambda_{a_k} t^{a_1, \dots, a_k}_{a_1, \dots, a_k}(u),$$

where $t^{a_1, \dots, a_k}_{b_1, \dots, b_k} = \sum_{\sigma \in S_k} (-1)^\sigma \cdot t_{a_{\sigma(1)} b_1}(u) \dots t_{a_{\sigma(k)} b_k}(u - k + 1)$ is quantum minor of $T(u)$.

Bethe subalgebras

Theorem (Nazarov, Olshanski, 1996)

Suppose that $C \in \mathfrak{h}^{reg}$. Then Bethe subalgebra $B(C)$ is free and maximal commutative. The coefficients of the series $\tau_k(u, C)$ are free generators for $B(C)$.

Limit subalgebras

$$\deg t_{ij}^{(r)} = r$$

$$B_r(C) := Y_r(gl_n) \cap B(C)$$

$$\theta_r : \mathfrak{h}^{reg} \to \bigtimes_{i=1}^{r} \mathrm{Gr}(d(i), \dim Y_i(gl_n)), C \to (B_1(C), \dots, B_r(C)).$$

Denote the closure of $\theta_r(\mathfrak{h}^{reg})$ (with respect to Zariski topology) by Z_r. We have natural projections

$$\rho_k : Z_r \to Z_{r-1}.$$

Let us define inverse limit $Z = \varprojlim \rho_k$. Z naturally parameterizes some new commutative subalgebras with the same Poincare series, called limit Bethe subalgebras.

Main theorem

Theorem

1) Z is a smooth algebraic variety isomorphic to $\overline{M_{0,n+2}}$.
2) For any point $X \in \overline{M_{0,n+2}}$, the corresponding subalgebra $B(X)$ in $Y(gl_n)$ is free and maximal commutative.

Description of limit algebras

How to think about $\overline{M_{0,n+2}}$?

The points of $\overline{M_{0,n+2}}$ are isomorphism classes of curves of genus 0, with $n+2$ ordered marked points and possibly with nodes, such that each component has at least 3 distinguished points (either marked points or nodes). Elements of $\overline{M_{0,n+2}}$ can be represented by pictures like the following on the right.

Conditions:

1. $n+2$ – marked points $0, z_1, \ldots, z_n, \infty$;
2. At least 3 marked points or nodes at every component;
3. Nodes are marked too.

Description of limit algebras

The limit Bethe subalgebra corresponding to the curve $X \in \overline{M_{0,n+2}}$ is the tensor product of the following 3 commuting subalgebras:

$$i(B(C)) \otimes_{\mathbb{C}} \psi(B(X_\infty)) \otimes_{ZU(\bigoplus_{\lambda \neq 0} \mathfrak{gl}_{k_\lambda})} \hat{F}(X_\lambda)$$

Here i and ψ some embedding of corresponding Yangians to $Y(\mathfrak{gl}_n)$, C some diagonal matrix, $\hat{F}(X_\lambda)$ – shift of argument subalgebras of $U(gl_n)$ corresponding to X_λ.

Conjecture

If C is real, then $B(C)$ acts with simple spectrum on certain class of finite-dimensional representations of $Y(\mathfrak{gl}_n)$.

Yangian for $\mathfrak{g}$

Let $\mathfrak{g}$ be an arbitrary complex simple Lie algebra. Due to Drinfeld, there exists so-called pseudo-uneversal R-matrix $\mathcal{R}(u)$. Suppose we have any finite-dimensional representation $\rho : Y(\mathfrak{g}) \to \mathrm{End}\,(V)$ (not a sum of trivial). Evaluate $R(u) = (\rho \otimes \rho)\mathcal{R}(-u)$.

Definition

Extended Yangian $X(\mathfrak{g})$ *for* $\mathfrak{g}$ is a complex unital associative algebra with countably many generators $t_{ij}^{(1)}, t_{ij}^{(2)}, \ldots$ where $1 \leqslant i, j \leqslant \dim V$, and the defining relations

$$R(u-v)T_1(u)T_2(v) = T_1(u)T_2(v)R(u-v).$$

where $T(u) = (t_{ij}(u))_{i,j=1}^{\dim V}$,

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)}u^{-1} + t_{ij}^{(2)}u^{-2} + \ldots \in X(\mathfrak{g})[[u^{-1}]].$$

Yangian for $\mathfrak{g}$

Definition

Yangian $Y(\mathfrak{g})$ *for* $\mathfrak{g}$ is defined as factor of $X(\mathfrak{g})$ by some relation $Z(u) = 1$, where $Z(u) \in X(\mathfrak{g}) \otimes End(V)[[u^{-1}]]$.

Wendlandt proved that this definition is correct, i.e. does not depends on representation V.
In fact in the same work it was proven that $X(\mathfrak{g}) \simeq Z(X(\mathfrak{g})) \otimes Y(\mathfrak{g})$.

Bethe subalgebras

Let $V = \oplus_i V(\omega_i, a_i)$ – sum of fundamental representations of $Y(\mathfrak{g})$.

Definition

Let $C \in G$. For any $1 \leqslant k \leqslant n$ introduce the series with coefficients in $Y(\mathfrak{g})$ by

$$\tau_k(u, C) = \operatorname{tr}_{V_{\omega_i}} \rho_i(C) T^i(u).$$

We call the subalgebra generated by the coefficients of $\tau_k(u, C)$ *Bethe subalgebra* and denote it by $B(C)$.