

(Finite) W -algebras associated to truncated current Lie algebras

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June 4, 2018

Interactions of quantum affine algebras with cluster algebras, current algebras and categorification

A conference celebrating the 60th birthday of Prof. Vyjayanthi Chari

The Catholic University of America

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A brief review of W-algebras in the literature

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Notation

- ▶ \mathbb{Z} : integers, \mathbb{C} : complex numbers.
- ▶ \otimes over \mathbb{C} unless explicit mention.
- ▶ \mathfrak{g} : fin.dim. s.s. Lie algebra over \mathbb{C} .
- ▶ $e \in \mathfrak{g}$ nonzero nilpotent element.
- ▶ $\{e, f, h\} \subset \mathfrak{g}$ an \mathfrak{sl}_2 -triple:

$$[e, f] = h, [h, e] = 2e, [h, f] = -2f.$$

- ▶ We fix a non-degenerate invariant bilinear form $(\cdot | \cdot)$ on \mathfrak{g} . Invariance means $([x, y] | z) = (x | [y, z]) \quad \forall x, y, z \in \mathfrak{g}$.

Nilpotent element and good \mathbb{Z} -grading

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Definition

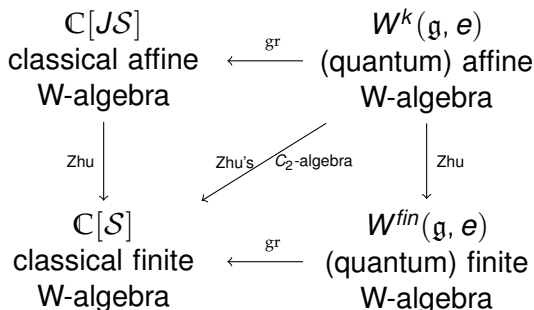
A \mathbb{Z} -grading $\Gamma : \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ is called **good** if $\exists e \in \mathfrak{g}(2)$, s.t. the above condition (\star) is satisfied, and e is called a **good element** w.r.t. Γ . We call Γ **even** if $\mathfrak{g}(i) = 0 \forall$ odd i .

Various W-algebras

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S is a Poisson variety sitting in \mathfrak{g}^* , and JS the arc space of S .

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Truncated current Lie algebras (TCLA)

The *level p truncated current Lie algebra* associated to \mathfrak{g} is

$$\mathfrak{g}_p := \mathfrak{g} \otimes \left(\mathbb{C}[t]/t^{p+1}\mathbb{C}[t] \right),$$

with Lie bracket: $[a \otimes t^i, b \otimes t^j] = [a, b] \otimes t^{i+j}$, where $t^{i+j} = 0$ when $i + j > p$.

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Lemma

$(\cdot \mid \cdot)_p$ is a non-degenerate invariant sym. bilinear form on \mathfrak{g}_p .

Preparation for finite W-algebras

Given a good \mathbb{Z} -grading $\Gamma : \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ with good element e , there exists $h_\Gamma \in \mathfrak{g}$, s.t. $[h_\Gamma, y] = iy, \forall y \in \mathfrak{g}(i)$.

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$$\mathfrak{m}_{\mathfrak{l}, \rho} = \left(\bigoplus_{i \leq -2} \mathfrak{g}_\rho(i) \right) \oplus \mathfrak{l}_\rho.$$

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Definition (via Whittaker model)

The finite W-algebra associated to (\mathfrak{g}_p, e) is defined to be the associative algebra H_{χ_p} with the multiplication

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Remark There are other equivalent definitions.

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There is a filtration $\{K_n H_{\chi_p}\}_{n \geq 0}$ on H_{χ_p} s.t. the associated graded $\text{gr}_K H_{\chi_p}$ is isomorphic to $\mathbb{C}[\mathcal{S}_{\chi_p}]$ as Poisson algebras.

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Theorem (Truncated current version of Kostant's theorem)

Let e be principal. Then the finite W -algebra H_{χ_p} associated to (\mathfrak{g}_p, e) is isomorphic to the center of $U(\mathfrak{g}_p)$.

Whittaker modules and Skryabin equivalence

Definition (Whitt.=Whittaker)

A \mathfrak{g}_p -module M is called a *Whitt. module* if $a - \chi_p(a)$ acts loc. nil. on M , $\forall a \in \mathfrak{m}_{\iota,p}$.

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- (2) Let $N \in H_{\chi_p}\text{-Mod}$. Then $Q_{\chi_p} \otimes_{H_{\chi_p}} N \in \mathfrak{g}_p\text{-Wmod}^{\chi_p}$.

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- (2) Let $N \in H_{\chi_p}\text{-Mod}$. Then $Q_{\chi_p} \otimes_{H_{\chi_p}} N \in \mathfrak{g}_p\text{-Wmod}^{\chi_p}$.
- (3) $\text{Wh}(-)$ and $Q_{\chi_p} \otimes_{H_{\chi_p}} -$ give an equivalence of categories between $\mathfrak{g}_p\text{-Wmod}^{\chi_p}$ and $H_{\chi_p}\text{-Mod}$.

Two remarks

- ▶ Consider more generally truncated multicurrent algebras, i.e., Lie algebras of the following form,

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- ▶ Affine W -algebras associated to TCLAs can be defined as in the semisimple case, they are vertex algebras and their Zhu algebras are finite W -algebras associated to TCLAs. More properties of these affine W -algebras are still under study.

Thanks !