(Finite) W-algebras associated to truncated current Lie algebras

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Interactions of quantum affine algebras with
cluster algebras, current algebras and categorification

A conference celebrating
the 60^{th} birthday of Prof. Vyjayanthi Chari

The Catholic University of America

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A brief review of W-algebras in the literature

Truncated current Lie algebras

Finite W-algebras associated to truncated current Lie algebras

Notation

- $\mathbb{Z}$: integers, $\mathbb{C}$: complex numbers.
- $\otimes$ over $\mathbb{C}$ unless explicit mention.
- $\mathfrak{g}$: fin.dim. s.s. Lie algebra over $\mathbb{C}$.
- $e \in \mathfrak{g}$ nonzero nilpotent element.
- $\{e, f, h\} \subset \mathfrak{g}$ an $\mathfrak{sl}_2$-triple:

$$[e, f] = h, [h, e] = 2e, [h, f] = -2f.$$

- We fix a non-degenerate invariant bilinear form $(\cdot \mid \cdot)$ on $\mathfrak{g}$. Invariance means $([x, y] \mid z) = (x \mid [y, z]) \quad \forall x, y, z \in \mathfrak{g}$.

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Definition

A $\mathbb{Z}$-grading $\Gamma : \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ is called good if $\exists\, e \in \mathfrak{g}(2)$, s.t. the above condition $(\star)$ is satisfied, and e is called a good element w.r.t. Γ. We call Γ even if $\mathfrak{g}(i) = 0 \; \forall$ odd i.

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$\mathcal{S}$ is a Poisson variety sitting in $\mathfrak{g}^*$, and $J\mathcal{S}$ the arc space of $\mathcal{S}$.

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$$\mathfrak{g}_p := \mathfrak{g} \otimes \left(\mathbb{C}[t] / t^{p+1} \mathbb{C}[t] \right),$$

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Given a good $\mathbb{Z}$-grading $\Gamma : \mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$ with good element e, there exists $h_\Gamma \in \mathfrak{g}$, s.t. $[h_\Gamma, y] = iy, \forall\, y \in \mathfrak{g}(i)$.

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Definition (via Whittaker model)

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Remark There are other equivalent definitions.

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Theorem (Quantization of jets of Slodowy slice)

There is a filtration $\{K_n H_{\chi_p}\}_{n \geq 0}$ on H_{χ_p} s.t. the associated graded $\mathrm{gr}_K H_{\chi_p}$ is isomorphic to $\mathbb{C}[\mathcal{S}_{\chi_p}]$ as Poisson algebras.

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Theorem (Truncated current version of Kostant's theorem)

Let e be principal. Then the finite W-algebra H_{χ_p} associated to $(\mathfrak{g}_p, e)$ is isomorphic to the center of $U(\mathfrak{g}_p)$.

Whittaker modules and Skryabin equivalence

Definition (Whitt.=Whittaker)

A $\mathfrak{g}_p$-module M is called a *Whitt. module* if $a - \chi_p(a)$ acts loc. nil. on M, $\forall\, a \in \mathfrak{m}_{\mathfrak{l},p}$.

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(2) *Let $N \in H_{\chi_p}$-Mod. Then $Q_{\chi_p} \otimes_{H_{\chi_p}} N \in \mathfrak{g}_p$-Wmod$^{\chi_p}$.*

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Let $\mathfrak{g}_p$-Wmod$^{\chi_p}$ be the category of f.g. Whittaker $\mathfrak{g}_p$-modules and H_{χ_p}-Mod the category of f.g. left H_{χ_p}-modules.

Theorem (Skryabin equivalence for TCLA)

(1) *Let $M \in \mathfrak{g}_p$-Wmod$^{\chi_p}$. Then $\mathrm{Wh}(M) \in H_{\chi_p}$-Mod.*

(2) *Let $N \in H_{\chi_p}$-Mod. Then $Q_{\chi_p} \otimes_{H_{\chi_p}} N \in \mathfrak{g}_p$-Wmod$^{\chi_p}$.*

(3) *$\mathrm{Wh}(-)$ and $Q_{\chi_p} \otimes_{H_{\chi_p}} -$ give an equivalence of categories between $\mathfrak{g}_p$-Wmod$^{\chi_p}$ and H_{χ_p}-Mod.*

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- Affine W-algebras associated to TCLAs can be defined as in the semisimple case, they are vertex algebras and their Zhu algebras are finite W-algebras associated to TCLAs. More properties of these affine W-algebras are still under study.

Thanks !