Branching laws for non-generic representations

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Main Problem

- F - a p-adic field. π - irreducible smooth representation of $GL_n(F)$.
- How does $\pi|_{GL_{n-1}(F)}$ decompose?
- What are the irreducible quotients of $\pi|_{GL_{n-1}(F)}$?
- It is long-known (Jacquet - Piatetski-Shapiro - Shalika) that

$$\mathrm{Hom}(\pi_1|_{GL_{n-1}(F)}, \pi_2) \neq 0 ,$$

for all *generic* irreducible π_1, π_2.
- **Goal:** What can be said about the non-generic case?One would like to describe the pairs (π_1, π_2) for which the Hom space above is non-zero.

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Tools: Parabolic induction

Representations of $\{GL_n(F)\}_n$ have an intriguing product structure.

Given $\pi_1 \in \mathrm{Rep}\, GL_{n_1}(F)$ and $\pi_2 \in \mathrm{Rep}\, GL_{n_2}(F)$, we can think of $\pi_1 \otimes \pi_2$ as a representation of the parabolic subgroup

$$P = \begin{pmatrix} GL_{n_1}(F) & * \\ 0 & GL_{n_2}(F) \end{pmatrix} < GL_{n_1+n_2}(F) \,.$$

$$\pi_1 \times \pi_2 := ind_P^{GL_{n_1+n_2}(F)}(\pi_1 \otimes \pi_2) \in \mathrm{Rep}\, GL_{n_1+n_2}(F)$$

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Tools: Derivatives

For every $0 \leqslant i \leqslant n$, there are well-defined functors

$$\begin{array}{ccc} \text{Rep } GL_n(F) & \to & \text{Rep } GL_{n-i}(F) \\ \pi & \mapsto & \pi^{(i)} \\ \pi & \mapsto & {}^{(i)}\pi \end{array}$$

called left and right Bernstein-Zelevinski derivatives. Derivatives of an irreducible representation are objects of finite length.

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Filtration

$$\pi \in \operatorname{Irr} GL_n(F)$$

Classical results of Bernstein-Zelevinski give a filtration of the $GL_{n-1}(F)$-representation $(\pi|_{GL_{n-1}(F)}, U)$

$$\{0\} = U_{n+1} \subset U_n \subset \cdots \subset U_1 = U ,$$

so that each U_i/U_{i+1} is well understood.

A question about morphisms

$$\mathrm{Hom}(\pi_1|_{GL_{n-1}(F)}, \pi_2) \neq 0$$

$$\Longrightarrow \quad \mathrm{Hom}\left(U_i/U_{i+1}, \pi_2\right) \neq 0 \ ,$$

for some i. From an application of Frobenius reciprocity (a certain adjunction of functors), we can rewrite the above Hom space in another form and obtain:

$$\mathrm{Hom}_{GL_{n-i}(F)}\left(|\det|_F^{1/2} \otimes \pi_1^{(i)}, \ {}^{(i-1)}\pi_2\right) \neq 0 \ .$$

Now, we are asking about Hom spaces of finite-length representations!

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Nice classes of representations

- Giving a branching law for all irreducible representations is considered a wild problem. Instead, we will focus on nice subclasses of representations.
- Motivated by harmonic analysis on automorphic spaces, Arthur attached unitarizable representations to certain (strict) *Arthur parameters*. We call them (strict) *Arthur-type* representations.

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Arthur-type representations

- **Arthur parameter - combinatorial description:** A collection of triples

$$\phi = \{(\rho_i, a_i, b_i)\}_{i\in I} \ ,$$

where $\rho_i \in \mathrm{Irr}\, GL_{d_i}(F)$ are supercuspidal representations, $a_i, b_i \in \mathbb{Z}_{>0}$.

- Each triple (ρ_i, a_i, b_i) defines a *Speh* representation $\pi_{\rho_i}^{a_i,b_i} \in \mathrm{Irr}\, GL_{d_i a_i b_i}(F)$.
- Such ϕ gives rise to

$$\pi(\phi) = \bigtimes_{i\in I} \pi_{\rho_i}^{a_i,b_i} \in \mathrm{Irr}\, GL_n(F) \ ,$$

where $n = \sum_{i\in I} d_i a_i b_i$.

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Gan-Gross-Prasad conjecture

Conjecture (Gan-Gross-Prasad)

For Arthur parameters ϕ_1, ϕ_2 so that $\pi(\phi_1) \in \operatorname{Irr} GL_n(F)$ and $\pi(\phi_2) \in \operatorname{Irr} GL_{n-1}(F)$,

$$\operatorname{Hom}\left(\pi(\phi_1)|_{GL_{n-1}(F)}, \pi(\phi_2)\right) \neq 0$$

holds, if and only if, we can write

$$\phi_1 = \{(\rho_i, a_i, b_i)\}_{i\in I}, \quad I = I^+ \cup I^-,$$

$$\phi_2 = \{(\rho_i, a_i, b_i - 1)\}_{i\in I^-} \cup \{(\rho_i, a_i, b_i + 1)\}_{i\in I^+} \cup \{(\rho_j, a_j, 1)\}_{j\in J}$$

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Results

Theorem (GGP)

Conjecture is true when $|I| = 1$ (i.e. π_1 is Speh representation) or when $a_i = 1$, $\forall i$.

Essential reason (for $|I| = 1$ case): Highest non-zero derivative of $\pi_\rho^{a,b}$ is $|\det|^{-1/2} \otimes \pi_\rho^{a,b-1}$.

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Results: One direction of GGP fully resolved

Theorem (G.)

If $\pi_1 = \pi(\phi_1) \in \mathrm{Irr}\, GL_n(F)$ and $\pi_2 = \pi(\phi_2) \in \mathrm{Irr}\, GL_{n-1}(F)$ are Arthur-type representations with

$$\mathrm{Hom}\left(\pi_1|_{GL_{n-1}(F)}, \pi_2\right) \neq 0 ,$$

then ϕ_1, ϕ_2 must comply with the conditions stated in the GGP conjecture.

A similar statement holds when π_1, π_2 are general unitarizable representations.

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A similar statement holds when π_1, π_2 are general unitarizable representations.

Results: Converse direction

Theorem (G.)

Conjecture is true when *one* of π_1, π_2 is generic.

"Where the dog lies buried"

- Derivatives of Speh representations $\left(\pi_\rho^{a,b}\right)^{(i)}$ are all either zero or a given irreducible representation, which we call *quasi-Speh*.
- For a given Arthur-type representation

$$\pi = \pi_{\rho_1}^{a_1,b_1} \times \cdots \times \pi_{\rho_k}^{a_k,b_k} ,$$

a derivative $\pi^{(i)}$ is built out of representations of the form

$$\left(\pi_{\rho_1}^{a_1,b_1}\right)^{(j_1)} \times \cdots \times \left(\pi_{\rho_k}^{a_k,b_k}\right)^{(j_k)} .$$

where $i = j_1 + \ldots + j_k$.

- Products of quasi-Speh representations are not irreducible !!

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- Products of quasi-Speh representations are not irreducible !!

- We would like to determine for which Arthur parameters there can be a non-zero morphism in

$$\mathrm{Hom}\left(|\det|_F^{1/2} \left(\pi_{\rho_1}^{a_1,b_1}\right)^{(j_1)} \times \cdots \times \left(\pi_{\rho_k}^{a_k,b_k}\right)^{(j_k)} , \right.$$

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- *If* we knew that non-zero elements of this Hom space factor through a *unique quotient* and a *unique sub-representation*, then the GGP rule would follow from a combinatorial argument.

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- *If* we knew that non-zero elements of this Hom space factor through a *unique quotient* and a *unique sub-representation*, then the GGP rule would follow from a combinatorial argument.

... and now for something completely different...

Affine Hecke algebras

- The category of smooth representation of $GL_n(F)$ can be decomposed as a product of abelian categories $\prod_\Theta \mathcal{M}_\Theta$, called *Bernstein blocks*.
- Each block $\mathcal{M}_\Theta$ is equivalent to the category of modules over a certain intertwiner algbera A_Θ.
- It was shown that A_Θ can always be described as tensor products of affine Hecke algebras associated to type A root data.
- Thus, our problem can be translated (with some care) into a problem on Hom spaces in categories of module over type A affine Hecke algebras.
- Parabolic induction product can be translated to algebra module induction.

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Quantum affine Schur-Weyl duality

- The classical Schur-Weyl duality can be thought of as an exact functor from module over the algebra $\mathbb{C}[S_n]$ into modules over the algebra $U(\mathfrak{sl}_N)$.
- This functor can by quantized to move from modules over the Hecke algebra of S_n to the quantum group $U_q(\mathfrak{sl}_N)$.
- The *quantum affine Schur-Weyl* duality functor $\mathcal{F}_{k,N}$ as defined by Chari-Pressley takes fin.-dim. modules over the affine Hecke algebra GL_k to fin.-dim. modules over $U_q(\hat{\mathfrak{sl}}_N)$.
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Dictionary

- A representation $\pi \in \operatorname{Irr} GL_n(F)$ is generic, if and only if, $\mathcal{F}_{n,2}(\pi) \neq 0$.
- Images under $\mathcal{F}_{k,N}$ of Speh representations are precisely Kirillov-Reshetekhin modules.
- Quantum affine Schur-Weyl duality pleasantly transforms the induction product into a tensor product, i.e.

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Highest weight modules

- There is a triangular decomposition $U_q(\hat{\mathfrak{sl}}_N) = \hat{U}^- \otimes \hat{U}^0 \otimes \hat{U}^+$, similar to the classical setting of universal enveloping algebras.
- For a module V over $U_q(\hat{\mathfrak{sl}}_N)$ (in the category of our interest), we say that $v \in V$ is a *highest weight vector*, if v is an eigenvector for the algebra $\hat{U}^0$ and $\hat{U}^+ \cdot v = 0$.
- A module is said to be a *highest weight* (or *cyclic*) module if it is spanned by a highest weight vector.
- Easy to see that highest weight modules have a unique irreducible quotient!

Highest weight modules

- There is a triangular decomposition $U_q(\hat{\mathfrak{sl}}_N) = \hat{U}^- \otimes \hat{U}^0 \otimes \hat{U}^+$, similar to the classical setting of universal enveloping algebras.
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- **Question:** Given k quasi-Speh representations, can they be numbered $\pi_1, \ldots, \pi_k$ so that the product of their images $V_i = \mathcal{F}_{n_i,N}(\pi_i)$

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Theorem (Hernandez)

For irreducible modules $V_1, \dots, V_k$ of $U_q(\hat{\mathfrak{sl}}_N)$, such that

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is highest weight for all $i < j$, the product

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Some ideas of proof of GGP conjectures

- By Hernandez, enough to show that $V_i \otimes V_j$ is highest weight, for all $i < j$.
- **Question reformulated:** Is there an order on Speh representations, such that when $\pi_1 < \pi_2$, their quasi-Speh derivatives $\sigma_1 = \pi_1^{(t_1)}, \sigma_2 = \pi_1^{(t_2)}$ would give a highest weight module $\mathcal{F}_{n,N}(\sigma_1 \times \sigma_2)$?
- Lapid-Minguez showed that the product of every pair of quasi-Speh (more generally, *ladder*) representations has a unique irreducible quotient. Moreover, they gave a combinatorial algorithm for computing the isomorphism class of that quotient.
- The image of $\sigma_1 \times \sigma_2$ is highest weight, if and only if, the quotient is given by the Langlands parameter which is the sum of Langlands parameter of σ_1, σ_2.

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Thank you !