Quantum exterior algebras extended by groups

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June 4, 2017

Group extensions of quantum exterior algebras

Let $\mathbb{K}$ be a field and V be a $\mathbb{K}$-vector space with basis $\{v_1, v_2, ..., v_n\}$. Let $\mathbf{q} = \{q_{ij} | q_{ij} \in \mathbb{K} - \{0\} \text{ and } q_{ij} = q_{ji}^{-1}\}$ be the set of quantum scalars.

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Definition

The group extension, $A \rtimes G$, is $A \otimes \mathbb{K}G$ as a vector space with multiplication

$$(vg)(wh) = v({}^g w)gh.$$

Quantum Drinfeld Hecke algebras

Let $\mathscr{H}_{q,\kappa} := T(V) \rtimes G/(v_j v_i + q_{ji} v_i v_j - \sum_{g \in G} \kappa_g(v_i, v_j)g)$.

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Define the $\mathscr{H}_{q,\kappa}$ for which the associated graded algebra is isomorphic to $S_q(V) \rtimes G$ to be a *quantum Drinfeld Hecke algebra over* $\mathbb{K}$.

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Theorem (Levandovskyy, Shepler)

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(i) G *acts on* $S_{\mathbf{q}}(V)$ *by automorphisms and* $q_{ij} = q_{ji}^{-1}$, $q_{ii} = 1$,

(ii) $\kappa(v_j, v_i) = -q_{ij}^{-1}\kappa(v_i, v_j)$,

(iii) $0 = (q_{ik}q_{jk}{}^h v_k - v_k)\kappa_h(v_i, v_j) + (q_{jk}v_j - q_{ij}{}^h v_j)\kappa_h(v_i, v_k) + ({}^h v_i - q_{ij}q_{ik}v_i)\kappa_h(v_j, v_k)$,

(iv) $\kappa_{h^{-1}gh}(v_r, v_s) = \sum_{i<j} det_{rsij}(h)\kappa_g(v_i, v_j)$,

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Definition

Let t be an indeterminant. A *deformation* of A over $\mathbb{K}[t]$ is an associative algebra $A[t]$ with multiplication given by

$$a * b = ab + \mu_1(a \otimes b)t + \mu_2(a \otimes b)t^2 + \ldots$$

for linear maps $\mu_i : A \otimes A \to A$.

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In order for the deformation to be associative, μ_1 must be a Hochschild 2-cocycle with $[\mu_1, \mu_1]$ a coboundary.

Translating to HH

Theorem (Naidu, Witherspoon)

The quantum Drinfeld Hecke algebras over $\mathbb{C}[t]$ are precisely the deformations of $S_q(V) \rtimes G$ over $\mathbb{C}[t]$ with $\deg \mu_i = -2i$ for all $i \geq 1$.

Theorem (Naidu, Witherspoon)

Assume that the action of G on V extends to an action on $\Lambda_q(V)$ and $S_q(V)$ by algebra automorphisms. Then each constant Hochschild 2-cocycle on $S_q(V) \rtimes G$ gives rise to a quantum Drinfeld Hecke algebra.

What happens with $\Lambda_q(V) \rtimes G$?

$$\Lambda_{\mathbf{q}}(V) := T(V)/(v_i v_j - q_{ji} v_j v_i, v_i^2).$$

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Let $\mathscr{H}_{q,\kappa,2} := T(V) \rtimes G[t]/(v_j v_i - q_{ji} v_i v_j - \sum_{g \in G} \kappa_g(v_i, v_j)g, v_i^2)$.

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Define the $\mathscr{H}_{q,\kappa,2}$ for which the associated graded algebra is isomorphic to $\Lambda_q(V) \rtimes G$ to be a *truncated quantum Drinfeld Hecke algebra over* $\mathbb{K}$.

What happens with $\Lambda_q(V) \rtimes G$?

Theorem (G, Uhl)

The factor algebra $\hat{\mathscr{H}}_{\mathbf{q},\kappa,2}$ is a truncated quantum Drinfeld Hecke algebra if and only if for all $g, h \in G$ and $1 \leq i < j < k \leq n$

(i) G *acts on* $\Lambda_{\mathbf{q}}(V)$ *by automorphisms and* $q_{ij} = q_{ji}^{-1}$,

(ii) $\kappa(v_j, v_i) = -q_{ij}^{-1}\kappa(v_i, v_j)$ *and* $\kappa(v_i, v_i) = 0$,

(iii) $0 = (q_{ik}q_{jk}{}^h v_k - v_k)\kappa_h(v_i, v_j) + (q_{jk}v_j - q_{ij}{}^h v_j)\kappa_h(v_i, v_k)$
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(v) $0 = q_{ij}v_i\kappa_h(v_i, v_j) + {}^h v_i\kappa_h(v_i, v_j)$ *and* $0 = q_{ij}{}^h v_j\kappa_h(v_i, v_j) + v_j\kappa_h(v_i, v_j)$,

(vi) $\sum_{i<j}(g_i^r g_j^r)\kappa_h(v_i, v_j) = 0$.

Example

Let $q_{12} = -1, q_{23} = \omega = q_{31}$ where $\omega = e^{\frac{2\pi i}{3}}$ and $G = \left\langle \begin{pmatrix} -\omega^2 & 0 & 0 \\ 0 & -\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$.

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$HH^2(\Lambda_q^3 \rtimes G) = span_k\{(v_1 v_2 I)\epsilon^*_{0,0,2}, (v_1 v_2 I)\epsilon^*_{1,1,0}, (v_1 v_3 I)\epsilon^*_{1,0,1},$
$(v_2 v_3 I)\epsilon^*_{0,1,1}, (I)\epsilon^*_{1,1,0}, (v_1 v_2 g^2)\epsilon^*_{0,0,2}, (g^2)\epsilon^*_{0,0,2},$
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All $\mu \in HH^2(\Lambda_q^3 \rtimes G)$ have $[\mu, \mu] = 0$.

From Hochschild cohomology

Theorem (G, Uhl)

The truncated quantum Drinfeld Hecke algebras over $\mathbb{K}[t]$ are the deformations of $\Lambda_{\mathbf{q}}(V) \rtimes G$ with polynomial deg $\mu_i = -2i$ for all $i > 0$ and $\mu_i(v_j, v_j) = 0$.

Theorem (G, Uhl)

If the G action on V extends to an action on $\Lambda_{\mathbf{q}}(V)$, then each constant Hochschild 2-cocyle of $\Lambda_{\mathbf{q}}(V) \rtimes G$ that sends $v_i \otimes v_i \mapsto 0$ for all $i \in \{1, 2, ..., n\}$ produces a truncated quantum Drinfeld Hecke algebra.

Example (revisited)

Let $q_{12} = -1, q_{23} = \omega = q_{31}$ where $\omega = e^{\frac{2\pi i}{3}}$ and $G = \left\langle \begin{pmatrix} -\omega^2 & 0 & 0 \\ 0 & -\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$.

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The only PBW deformation $\mathscr{H}_{q,\kappa,2}$ is a quotient of $T(V) \rtimes G[t]$ given by the relations

$$\begin{aligned} v_2 v_1 &= -v_1 v_2 + mI, \\ v_3 v_2 &= \omega v_2 v_3, \\ v_3 v_1 &= \omega^2 v_1 v_3, \text{ and} \\ v_i v_i &= 0 \end{aligned}$$

for $i \in \{1,2,3\}$ and $m \in k$.

Example

Let $G = \left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} = \{g, I\}$ and $q_{12} = -1$, $q_{13} = q_{23} = 1$.

$HH^2(\Lambda_q^3 \rtimes G) = span_{\mathbb{K}}\{(I)\epsilon^*_{1,1,0}, (g)\epsilon^*_{1,0,1}, (g)\epsilon^*_{0,1,1}\}$.

The PBW deformation $\mathscr{H}_{q,\kappa,2}$ is a quotient of $T(V) \rtimes G[t]$ given by the relations

$$\begin{aligned} v_2 v_1 &= -v_1 v_2 + m_1 I, \\ v_3 v_1 &= v_1 v_3 + m_2 g, \\ v_3 v_2 &= v_2 v_3 + m_3 g, \\ v_i^2 &= 0 \text{ for } i = 1, 2, 3 \end{aligned}$$

where $m_1, m_2, m_3 \in \mathbb{K}$.

Thank you!

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Proposition (G, Uhl)

In a truncated quantum Drinfeld Hecke algebra, only group elements that act diagonally on the vector space can support the parameter space.