Tensor Product Multiplicities via Upper Cluster Algebras

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Tensor Product Multiplicities

Let G be the connected, simply connected complex algebraic group of type Q. Let $V(\lambda)$ be the irreducible (finite-dimensional) representations of G of highest weight λ. The tensor product of two irreducible representations decomposes as

$$V(\mu) \otimes V(\nu) = \bigoplus_{\lambda \in P_+} c^{\lambda}_{\mu,\nu} V(\lambda).$$

To compute the multiplicity $c^{\lambda}_{\mu,\nu}$ is not easy.

The Algebra of Triple-tensor Invariants

We consider the algebra of triple-tensor invariants

$$\mathcal{A}_G := \left(k[G]^{U^-} \otimes k[G]^{U^-} \otimes k[G]^{U}\right)^G.$$

The algebra is multigraded by a triple of dominant weights (μ, ν, λ)

$$\bigoplus_{\lambda,\mu,\nu \in P_+} C^{\lambda}_{\mu,\nu},$$

with the $\mathbb{C}$-dimension of graded component $C^{\lambda}_{\mu,\nu}$ equal to $c^{\lambda}_{\mu,\nu}$.
It turns out that the algebra $\mathcal{A}_G$ is an upper cluster algebra!

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The Quiver Δ_Q^2

The data needed to define an upper cluster algebra is a seed (same as the cluster algebra). The ice quiver Δ_Q^2 in the initial seed can be constructed from the certain category related to Q (ADE quiver).

- The vertices of Δ_Q^2 are indecomposable projective presentations $P_+ \to P_-$;
- The arrows of Δ_Q^2 are irreducible morphisms and the AR-translations;
- The frozen vertices are $0 \to P_i$, $P_i \to 0$, and $P_i \xrightarrow{\mathrm{Id}} P_i$.

Theorem (Fei)

The algebra $\mathcal{A}_G$ is an upper cluster algebra $\overline{\mathcal{C}}(\Delta_Q^2)$.

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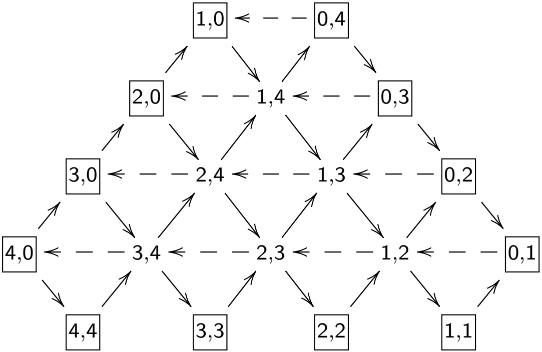
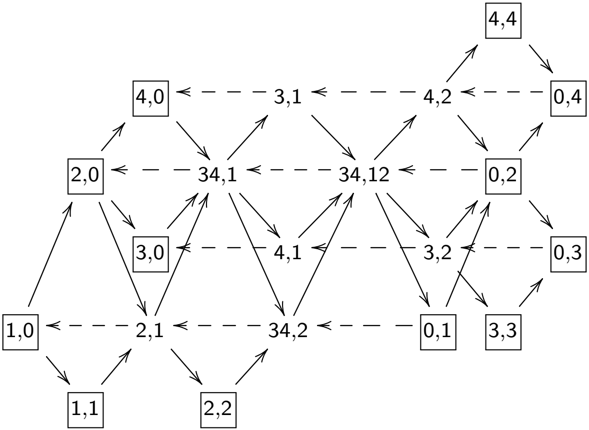
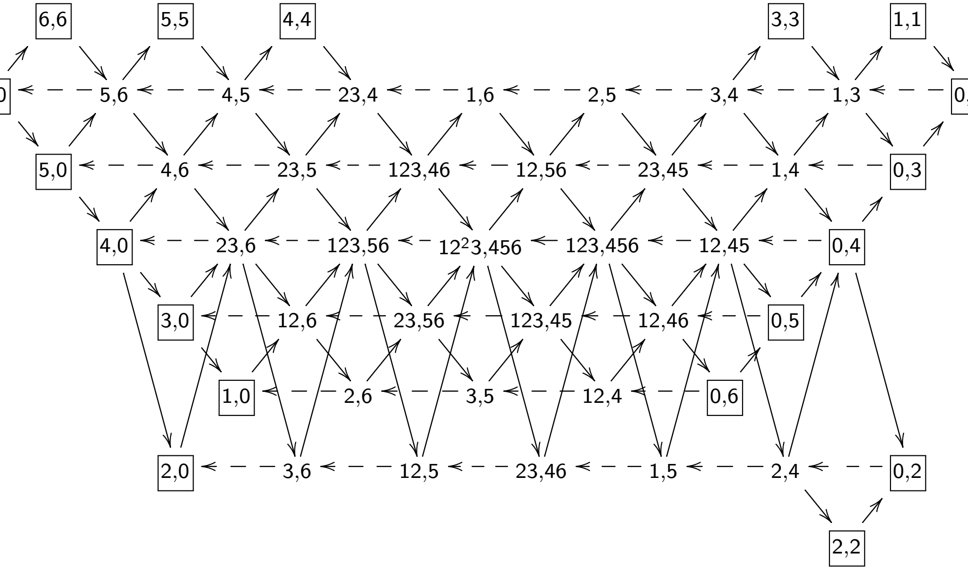
iARt A_4



iARt D_4



iARt E_6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2,6
3,5
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2,2

A Basis of $\mathcal{A}_G$

- There is a basis of $\mathcal{A}_G$ parametrized by μ-supported g-vectors.
- All such g-vectors lies in a cone, which can be explicitly described by the representation theory of Δ_Q^2.
- For each frozen vertex v of Δ_Q^2, there is an associated boundary representation T_v.
- The cone has a hyperplane presentation $\{x \in \mathbb{R}^{(\Delta_Q^2)_0} \mid Hx \geq 0\}$ where the rows of the matrix H are given by the dimension vectors of subrepresentations of T_v's.

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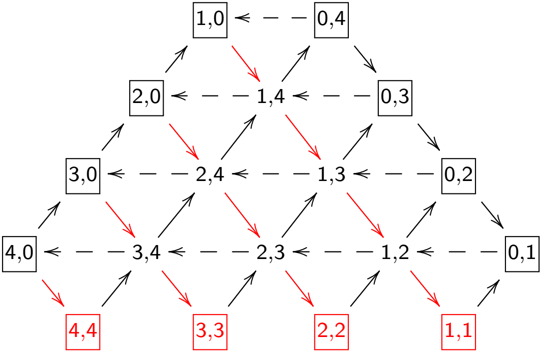
Boundary Representations in iARt A_4



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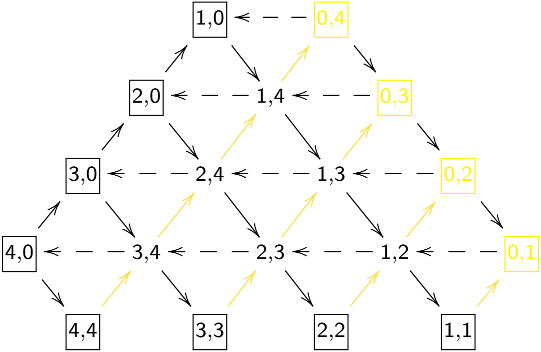

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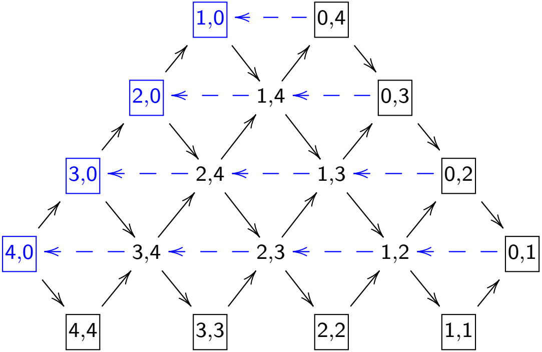
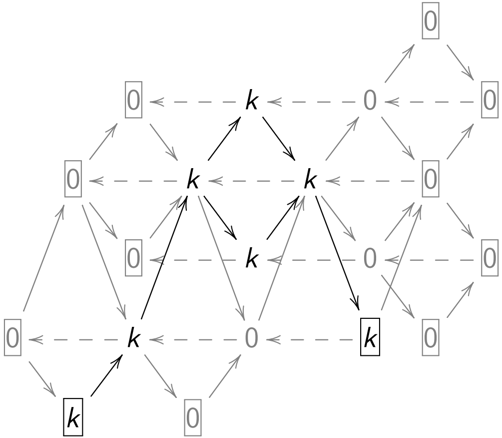


A Boundary Representation in iARt D_4



The Generic Cluster Character

Given a vector $\mathrm{g} \in \mathbb{Z}^{(\Delta_Q^2)_0}$, we can associate a generic representation $M := \mathrm{Coker}(\mathrm{g})$ of (Δ_Q^2, W_Q^2). The generic character CC maps μ-supported g-vectors (lattice points in the cone) to the upper cluster algebra.

$$CC(\mathrm{g}) = \mathbf{x}^{\mathrm{g}} \sum_{\mathrm{e}} \chi\big(\mathrm{Gr}^{\mathrm{e}}(M)\big)\mathbf{y}^{\mathrm{e}}.$$

Theorem (Fei)

The generic cluster character maps the lattice points in the cone to a basis of $\mathcal{A}_G$.

Corollary

The multiplicity $c_{\mu,\nu}^{\lambda}$ is counted by lattice points in the fibre polytope of the cone defined by (μ, ν, λ).

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Thank you

Happy Birthday to Professor Chari!