Cluster algebras and snake modules

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Cluster algebras

Cluster algebras were introduced by Fomin and Zelevinsky around 2000.

Example

$Q: \quad 1 \to 2$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{tikzcd}
 & \Big((x_1, x_2),\quad 1\to 2\Big) \arrow[dl, dash, "\mu_1"'] \arrow[dr, dash, "\mu_2"] & \\
\Big(\big(\frac{1+x_2}{x_1}, x_2\big),\quad 1\leftarrow 2\Big) \arrow[d, dash, "\mu_2"'] & & \Big(\big(x_1, \frac{x_1+1}{x_2}\big),\quad 1\leftarrow 2\Big) \arrow[d, dash, "\mu_1"] \\
\Big(\big(\frac{1+x_2}{x_1}, \frac{x_1+1+x_2}{x_1 x_2}\big),\quad 1\to 2\Big) \arrow[dr, dash, "\mu_1"'] & & \Big(\big(\frac{x_1+1+x_2}{x_1 x_2}, \frac{x_1+1}{x_2}\big)),\quad 1\to 2\Big) \arrow[dl, phantom, "\sim"] \\
 & \Big(\big(\frac{x_1+1}{x_2}, \frac{x_1+1+x_2}{x_1 x_2}\big),\quad 1\leftarrow 2\Big) &
\end{tikzcd}$$

$\mathcal{A}(Q) = \mathbb{Z}[x_1, x_2, \frac{1+x_2}{x_1}, \frac{x_1+1}{x_2}, \frac{x_1+x_2+1}{x_1 x_2}] \subseteq \mathbb{Q}(x_1, x_2)$

Representations of quantum affine algebras

- $\mathfrak{g}$ simple Lie algebra over $\mathbb{C}$
- $I = \{1, 2, \ldots, n\}$, where n is the rank of $\mathfrak{g}$
- $U_q(\hat{\mathfrak{g}})$ (untwisted) quantum affine algebra with quantum parameter $q \in \mathbb{C}^*$ not a root of unity

Chari and Pressley classified the finite dimensional simple $U_q(\hat{\mathfrak{g}})$-modules in terms of Drinfeld polynomials, i.e., I-tuples of polynomials $(P_1, \cdots, P_n)$, where P_i, $i = 1, \ldots, n$, are polynomials in one indeterminate u with coefficients in $\mathbb{C}$ and constant term 1.

For example, in A_n,

- $(1,\ldots,1,\underset{i}{1-au},1\ldots,1)$ is a Drinfeld polynomial of a fundamental module
- $(1,\ldots,1,\underset{i}{(1-au)(1-aq^2u)},1\ldots,1)$ is a Drinfeld polynomial of a Kirillov-Reshetikhin module
- $(1,\ldots,1,\underset{i}{(1-au)(1-aq^2u)},\underset{i+1}{(1-aq^5u)(1-aq^7u)},1,\ldots,1)$ is a Drinfeld polynomial of a minimal affinization.

Minimal affinizations were introduced by Chari in 1995.

Let $\mathcal{P}$ be the free abelian multiplicative group of monomials in infinitely many formal variables $(Y_{i,a})_{i\in I, a\in\mathbb{C}^*}$ and $\mathcal{P}^+ \subseteq \mathcal{P}$ denote the set of all dominant monomials. Then

$$\begin{aligned} \mathcal{P}^+ &\longleftrightarrow \{\text{f. d. simple } U_q(\hat{\mathfrak{g}})\text{-modules}\} \\ m &\longmapsto L(m). \end{aligned}$$

Example, in A_n,

Drinfeld polynomials	Dominant monomials	simple $U_q(\hat{\mathfrak{g}})$-modules
$(1,\ldots,1,\underset{i}{1-au},1\ldots,1)$	$Y_{i,a}$	$L(Y_{i,a})$
$(1,\ldots,1,\underset{i}{(1-au)(1-aq^2u)},1\ldots,1)$	$Y_{i,a}Y_{i,aq^2}$	$L(Y_{i,a}Y_{i,aq^2})$
$(1,\ldots,1,\underset{i}{(1-au)(1-aq^2u)},\underset{i+1}{(1-aq^5u)(1-aq^7u)},1,\ldots,1)$	$Y_{i,a}Y_{i,aq^2}Y_{i+1,aq^5}Y_{i+1,aq^7}$	$L(Y_{i,a}Y_{i,aq^2}Y_{i+1,aq^5}Y_{i+1,aq^7})$
	$Y_{1,aq^{-13}}Y_{2,aq^{-10}}Y_{3,aq^{-7}}Y_{2,aq^{-4}}Y_{1,aq^{-1}}$	$L(Y_{1,aq^{-13}}Y_{2,aq^{-10}}Y_{3,aq^{-7}}Y_{2,aq^{-4}}Y_{1,aq^{-1}})$

Snake modules were introduced by Mukhin and Young in 2012. Snake modules corresponds to ladder representations in the setting of representations of p-adic groups. Ladder representations were introduced by Lapid and Minguez in 2014.

Kirillov–Reshetikhin modules $\subset$ minimal affinizations $\subset$ prime snake modules $\subset$ snake modules

Frenkel and Reshetikhin introduced the theory of q-characters. The q-character is an injective ring homomorphism from Grothendick ring of the category of f. d. simple $U_q(\hat{\mathfrak{g}})$-modules to Laurent polynomial ring $\mathbb{Z}[Y_{i,a}^{\pm}]_{i\in I, a\in\mathbb{C}^*}$. For example, $\mathfrak{g} = \mathfrak{sl}_2$,

$$\chi_q(L(Y_{1,a})) = Y_{1,a} + Y_{1,aq^2}^{-1}.$$

Frenkel-Mukhin algorithm can be used to compute the q-characters of a large family of modules. For example, the modules whose q-characters have only one dominant monomial.

The simple module

$$L(Y_{i_T,aq^{k_T}} \cdots Y_{i_2,aq^{k_2}} Y_{i_1,aq^{k_1}})$$

is called a **snake module/prime snake module/minimal snake module** if for all $2 \leq t \leq T$, the point (i_t, k_t) is in **snake position/prime snake position/minimal snake position** with respect to (i_{t-1}, k_{t-1}).

A point (i', k') is said to be in **snake position** (respectively, **prime snake position**) with respect to (i, k) if

Type A_n : $k' - k \geq |i' - i| + 2$ and $k' - k \equiv |i' - i| \pmod 2$
(respectively, $\min\{2n + 2 - i - i', i + i'\} \geq k' - k \geq |i' - i| + 2$ and $k' - k \equiv |i' - i| \pmod 2$).

The point (i', k') is in **minimal snake position** to (i, k) if $k' - k$ is equal to the given lower bound.

Motivations

- Hernandez-Leclerc Conjecture: a certain subcategory $\mathcal{C}_\ell$ of the category of all finite dimensional $U_q(\hat{\mathfrak{g}})$-modules has a cluster algebra structure. In summer school, Leclerc introduced the conjecture and listed known results.
- The fact that prime snake modules are prime was proved by Mukhin and Young.

Our results

We proved the following result.

Theorem

Prime snake modules of type A_n, B_n are real.

Idea of the proof: we use "path description of q-characters" introduced by Mukhin and Young in 2012.

Remark

Prime snake modules of type A_n are real also follows from a recent result of Lapid and Minguez in 2017. They proved (in the language of representations of p-adic groups) that a large family of $U_q(\hat{\mathfrak{g}})$-modules (containing prime snake modules) of type A_n are real.

Our main results are

Theorem

(1) *Prime snake modules can be obtained using a sequence of mutations from the initial cluster of the cluster algebras constructed by Hernandez and Leclerc.*

(2) *We obtained a system of equations satisfied by q-characters of prime snake modules. The system of equations contains all prime snake modules and only contains prime snake modules. It is a natural generalization of T-systems.*

I will show how to find mutation sequences.

Mutation sequences and some examples

For every finite dimensional Lie algebra $\mathfrak{g}$, Hernandez and Leclerc constructed a cluster algebra with an infinite quiver Q, for example, in type A_5, see Figure 1.

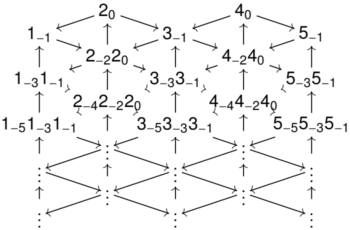
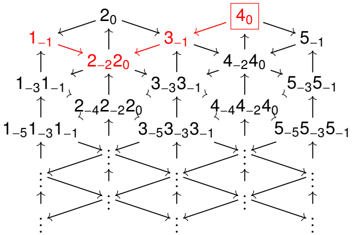


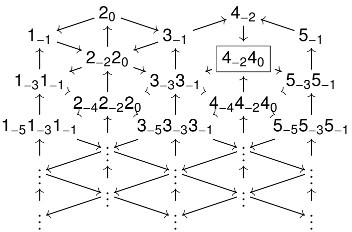
Figure: Quiver A_5

We denote $4_{-2}4_0 = L(Y_{4,aq^{-2}}Y_{4,a})$.

How to get

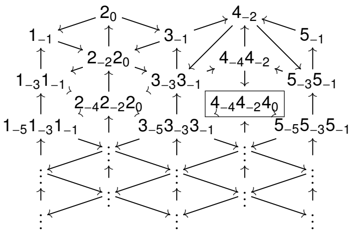
- $L(2_{-4}2_0)$, it is a non-minimal prime snake module, and
- $L(3_{-7}2_{-4}2_0)$.

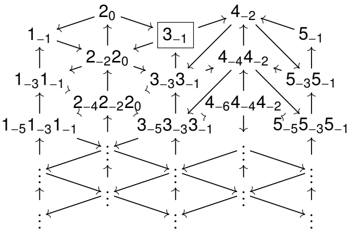




$$[4_0][4_{-2}] = [4_{-2}4_0] + [3_{-1}][5_{-1}]$$

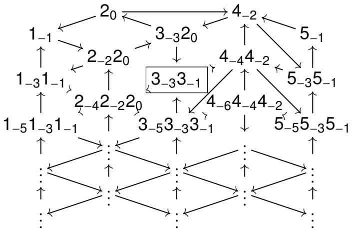
$$[4_{-2}4_0][4_{-4}4_{-2}] = [4_{-2}][4_{-4}4_{-2}4_0] + [3_{-3}3_{-1}][5_{-3}5_{-1}]$$

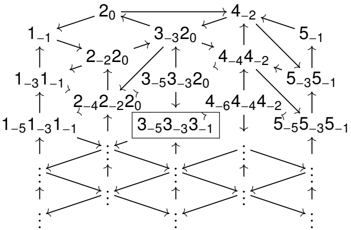




$$\begin{aligned}[4_{-4}4_{-2}4_0][4_{-6}4_{-4}4_{-2}] &= [4_{-4}4_{-2}][4_{-6}4_{-4}4_{-2}4_0] \\ &\quad + [3_{-5}3_{-3}3_{-1}][5_{-5}5_{-3}5_{-1}]\end{aligned}$$

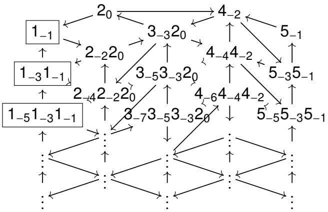
$$[3_{-1}][3_{-3}2_0] = [2_0][3_{-3}3_{-1}] + [2_{-2}2_0][4_{-2}]$$

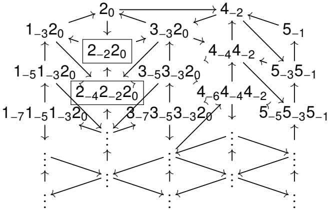




$$[3_{-3}3_{-1}][3_{-5}3_{-3}2_0] = [3_{-3}2_0][3_{-5}3_{-3}3_{-1}] + [2_{-4}2_{-2}2_0][4_{-4}4_{-2}]$$

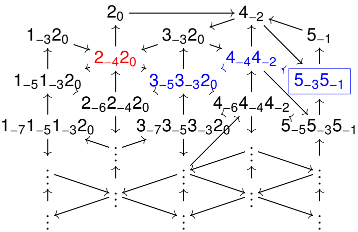
$$\begin{aligned}[3_{-5}3_{-3}3_{-1}][3_{-7}3_{-5}3_{-3}2_0] &= [3_{-5}3_{-3}2_0][3_{-7}3_{-5}3_{-3}3_{-1}] \\ &+ [2_{-6}2_{-4}2_{-2}2_0][4_{-6}4_{-4}4_{-2}]\end{aligned}$$

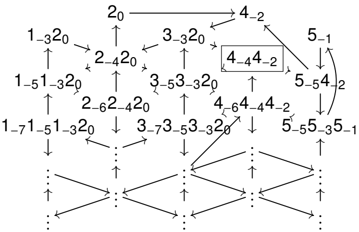




$[1_{-1}][1_{-3}2_0] = [2_0][1_{-3}1_{-1}] + [2_{-2}2_0]$,
$[1_{-3}1_{-1}][1_{-5}1_{-3}2_0] = [1_{-3}2_0][1_{-5}1_{-3}1_{-1}] + [2_{-4}2_{-2}2_0]$,
$[1_{-5}1_{-3}1_{-1}][1_{-7}1_{-5}1_{-3}2_0] = [1_{-5}1_{-3}2_0][1_{-7}1_{-5}1_{-3}1_{-1}] + [2_{-6}2_{-4}2_{-2}2_0]$

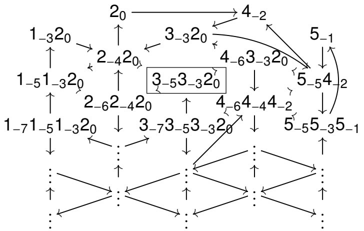
$[2_{-2}2_0][2_{-4}2_0] = [2_{-4}2_{-2}2_0][2_0] + [1_{-3}2_0][3_{-3}2_0]$,
$[2_{-4}2_{-2}2_0][2_{-6}2_{-4}2_0] = [2_{-4}2_0][2_{-6}2_{-4}2_{-2}2_0] + [1_{-5}1_{-3}2_0][3_{-5}3_{-3}2_0]$

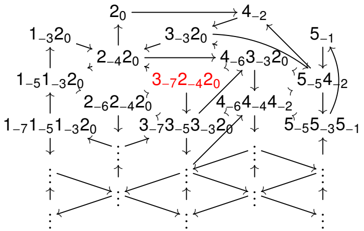




$$[5_{-3}5_{-1}][5_{-5}4_{-2}] = [5_{-5}5_{-3}5_{-1}][4_{-2}] + [4_{-4}4_{-2}].$$

$$[4_{-4}4_{-2}][4_{-6}3_{-3}2_0] = [3_{-3}2_0][4_{-6}4_{-4}4_{-2}] + [5_{-5}4_{-2}][3_{-5}3_{-3}2_0]$$





$$[3_{-5}3_{-3}2_0][3_{-7}2_{-4}2_0] = [3_{-7}3_{-5}3_{-3}2_0][2_{-4}2_0] + [4_{-6}3_{-3}2_0][2_{-6}2_{-4}2_0].$$

S-systems

We obtained a series of exchange relations in the process of mutations. They form a system of equations which we called S**-systems**, which is a natural generalization of T-systems. These equations in S-systems are of the form

$$[S_1][S_2] = [S_3][S_4] + [S_5][S_6],$$

where S_i, $i = 1, \ldots, 6$, are certain prime snake modules. For example:

- $[L(3_{-5}3_{-3}2_0)][L(3_{-7}2_{-4}2_0)] = [L(3_{-7}3_{-5}3_{-3}2_0)][L(2_{-4}2_0)] + [L(4_{-6}3_{-3}2_0)][L(2_{-6}2_{-4}2_0)]$

Thank you !