Maximal Dominant Weights for Affine Lie Algebra Representations

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June 5, 2018

Outline

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2. Integrable Highest Weight Modules
3. $B_n^{(1)}, V(k\Lambda_0)$
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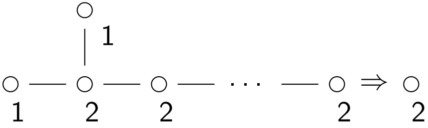
$\mathfrak{g} = B_n^{(1)}$

Consider $B_n^{(1)}$ with Cartan matrix and Dynkin diagram:

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & \dots & 0 \\ 0 & 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & \dots & 0 & 0 & -1 & 2 & -1 \\ 0 & \dots & 0 & 0 & 0 & -2 & 2 \end{pmatrix}$$



Definitions

- Let $\mathfrak{h} = \text{span}\{h_0, h_1, \ldots, h_n, d\}$ be the Cartan subalgebra
- $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_n\} \subset \mathfrak{h}^*$ is the set of simple roots. Note that $\alpha_j(h_i) = a_{ij}$.
- Let $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$ be the null root
- $c = h_0 + h_1 + 2h_2 + \cdots + 2h_{n-1} + h_n$ is the canonical central element
- The set $\{\Lambda_0, \Lambda_1, \ldots, \Lambda_n\}$ is the set of fundamental weights such that $\Lambda_j(h_i) = \delta_{ij}$ and $\Lambda_j(d) = 0$.

Integrable Highest Weight Modules

A weight λ is *dominant integral* if $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$ for $i \in I$. Let P^+ be the set of dominant integral weights.

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For $\Lambda \in P^+$ there exists an unique (up to isomorphism) irreducible, integrable highest weight module $V(\Lambda)$ generated by a highest weight vector v_Λ.

$V(\Lambda)$

Let $P(\Lambda)$ be the set of all weights of $V(\Lambda)$. If $\lambda \in P(\Lambda)$ then $\lambda = \Lambda - \sum_{i=0}^{n} b_i\alpha_i$ where $b_i \in \mathbb{Z}_{\geq 0}$.

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Definition

A weight $\lambda \in P(\Lambda)$ is called *maximal* if $\lambda + \delta \notin P(\Lambda)$. Denote the set of all maximal weights as $\max(\Lambda)$.

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Definition

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Then

$$P(\Lambda) = \bigcup_{\lambda \in \max(\Lambda)} \{\lambda - n\delta | n \in \mathbb{Z}_{\geq 0}\}.$$

Any $\lambda \in \max(\Lambda)$ is W-conjugate to some $\mu \in \max(\Lambda) \cap P^+$, which is known to be a finite set. However, only partial results for an explicit description of this set are known.

Find $\max(\Lambda) \cap P^+$

$\theta = \delta - a_0\alpha_0$
$\overline{\lambda} = \lambda - \lambda(c)\Lambda_0 - (\lambda|\Lambda_0)\delta$
$kC_{af} \cap (\overline{\Lambda} + \overline{Q}) = \{\overline{\lambda} \in \mathring{\mathfrak{h}}_{\mathbb{R}}^* | \overline{\lambda}(h_i) \geq 0 \text{ for all } i \in \mathring{I}, (\overline{\lambda}|\theta) \leq k\}$

Proposition (Kac)

The map $\lambda \mapsto \overline{\lambda}$ defines a bijection from $\max(\Lambda) \cap P_+$ onto $kC_{af} \cap (\overline{\Lambda} + \overline{Q})$. In particular, the set of dominant maximal weights of $V(\Lambda)$ is finite.

We want to determine the maximal dominant weights for the integrable $B_n^{(1)}$-module $V(k\Lambda_0)$.

Example: $B_5^{(1)}$, $V(3\Lambda_0)$

$\lambda \in \max(3\Lambda_0) \cap P_+ \implies \lambda = 3\Lambda_0 - \sum_{i=0}^{5} b_i\alpha_i$

Example: $B_5^{(1)}, V(3\Lambda_0)$

$\lambda \in \max(3\Lambda_0) \cap P_+ \implies \lambda = 3\Lambda_0 - \sum_{i=0}^{5} b_i\alpha_i$

Then

$$\overline{\lambda} = 3\Lambda_0 - \sum_{i=0}^{5} b_i\alpha_i - (3\Lambda_0 - \sum_{i=0}^{5} b_i\alpha_i)(c)\Lambda_0 - (3\Lambda_0 - \sum_{i=0}^{5} b_i\alpha_i \mid \Lambda_0)\delta$$

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$$= -\sum_{i=0}^{5} b_i\alpha_i + b_0\delta$$

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$\lambda \in \max(3\Lambda_0) \cap P_+ \implies \lambda = 3\Lambda_0 - \sum_{i=0}^{5} b_i\alpha_i$

Then

$$\begin{aligned}\overline{\lambda} &= 3\Lambda_0 - \sum_{i=0}^{5} b_i\alpha_i - (3\Lambda_0 - \sum_{i=0}^{5} b_i\alpha_i)(c)\Lambda_0 - (3\Lambda_0 - \sum_{i=0}^{5} b_i\alpha_i \mid \Lambda_0)\delta \\ &= -\sum_{i=0}^{5} b_i\alpha_i + b_0\delta \\ &= -\sum_{i=0}^{5} b_i\alpha_i + b_0(\alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5)\end{aligned}$$

Example: $B_5^{(1)}, V(3\Lambda_0)$

$\lambda \in \max(3\Lambda_0) \cap P_+ \implies \lambda = 3\Lambda_0 - \sum_{i=0}^5 b_i\alpha_i$
Then

$$\begin{aligned}
\overline{\lambda} &= 3\Lambda_0 - \sum_{i=0}^{5} b_i\alpha_i - (3\Lambda_0 - \sum_{i=0}^{5} b_i\alpha_i)(c)\Lambda_0 - (3\Lambda_0 - \sum_{i=0}^{5} b_i\alpha_i \mid \Lambda_0)\delta \\
&= -\sum_{i=0}^{5} b_i\alpha_i + b_0\delta \\
&= -\sum_{i=0}^{5} b_i\alpha_i + b_0(\alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5) \\
&= (b_0 - b_1)\alpha_1 + (2b_0 - b_2)\alpha_2 + (2b_0 - b_3)\alpha_3 + (2b_0 - b_4)\alpha_4 \\
&\quad + (2b_0 - b_5)\alpha_5
\end{aligned}$$

Example: $B_5^{(1)}$, $V(3\Lambda_0)$

Let $x_1 = b_0 - b_1, x_i = 2b_0 - b_i$ for $i = 2, \ldots, 5$.

Example: $B_5^{(1)}$, $V(3\Lambda_0)$

Let $x_1 = b_0 - b_1, x_i = 2b_0 - b_i$ for $i = 2, \ldots, 5$.

$$\begin{aligned}
(\textstyle\sum_{i=1}^5 x_i\alpha_i)(h_1) &= & 2x_1 - x_2 &\geq 0 \\
(\textstyle\sum_{i=1}^5 x_i\alpha_i)(h_2) &= & -x_1 + 2x_2 - x_3 &\geq 0 \\
(\textstyle\sum_{i=1}^5 x_i\alpha_i)(h_3) &= & -x_2 + 2x_3 - x_4 &\geq 0 \\
(\textstyle\sum_{i=1}^5 x_i\alpha_i)(h_4) &= & -x_3 + 2x_4 - x_5 &\geq 0 \\
(\textstyle\sum_{i=1}^5 x_i\alpha_i)(h_5) &= & -2x_4 + 2x_5 &\geq 0 \\
(\textstyle\sum_{i=1}^5 x_i\alpha_i \mid \theta) &= & x_2 &\leq 3
\end{aligned} \implies
\begin{array}{c}
(0,0,0,0,0)\\ (1,1,1,1,1)\\ (1,2,2,2,2)\\ (1,2,3,3,3)\\ (1,2,3,4,4)\\ (1,2,3,4,5)\\ (2,2,2,2,2)\\ (2,3,3,3,3)\\ (2,3,4,4,4)\\ (2,3,4,5,5)\\ (2,3,4,5,6)\\ (3,3,3,3,3)
\end{array}$$

What are the weights?

Recall $\lambda = 3\Lambda_0 - \sum_{i=0}^{5} b_i\alpha_i$. Given an element of $kC_{af} \cap (\overline{\Lambda} + \overline{Q})$, $(x_1, x_2, x_3, x_4, x_5)$, we need to find the corresponding λ.

What are the weights?

Recall $\lambda = 3\Lambda_0 - \sum_{i=0}^{5} b_i\alpha_i$. Given an element of $kC_{af} \cap (\overline{\Lambda} + \overline{Q})$, $(x_1, x_2, x_3, x_4, x_5)$, we need to find the corresponding λ.

Example $(x = (2, 3, 4, 5, 5))$

$$\begin{array}{rcl} b_0 - b_1 = 2 & \Longrightarrow & b_0 \geq 2 \\ 2b_0 - b_2 = 3 & \Longrightarrow & b_0 \geq \frac{3}{2} \\ 2b_0 - b_3 = 4 & \Longrightarrow & b_0 \geq 2 \\ 2b_0 - b_4 = 5 & \Longrightarrow & b_0 \geq \frac{5}{2} \\ 2b_0 - b_5 = 5 & \Longrightarrow & b_0 \geq \frac{5}{2} \end{array} \quad \Longrightarrow \quad b_0 \geq 3$$

Assume $b_0 = 3 + r$ for $r \in \mathbb{Z}_{>0}$. Then

$$\begin{aligned} \lambda + \delta = & 3\Lambda_0 - (3 + r - 1)\alpha_0 - (1 + r - 1)\alpha_1 - (3 + 2r - 2)\alpha_2 \\ & - (2 + 2r - 2)\alpha_3 - (1 + 2r - 2)\alpha_4 - (1 + 2r - 2)\alpha_5 \end{aligned}$$

Therefore, $b_0 = 3$ and $\lambda = 3\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5$

$\max(3\Lambda_0) \cap P_+$

x_2	**x** vector	Element of $\max(3\Lambda_0) \cap P^+$
0	$(0, 0, 0, 0, 0)$	$3\Lambda_0$
1	$(1, 1, 1, 1, 1)$	$3\Lambda_0 - \alpha_0 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$
2	$(1, 2, 2, 2, 2)$	$3\Lambda_0 - \alpha_0$
2	$(1, 2, 3, 3, 3)$	$3\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$
2	$(1, 2, 3, 4, 4)$	$3\Lambda_0 - 2\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
2	$(1, 2, 3, 4, 5)$	$3\Lambda_0 - 3\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
2	$(2, 2, 2, 2, 2)$	$3\Lambda_0 - 2\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5$
3	$(2, 3, 3, 3, 3)$	$3\Lambda_0 - 2\alpha_0 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$
3	$(2, 3, 4, 4, 4)$	$3\Lambda_0 - 2\alpha_0 - \alpha_2$
3	$(2, 3, 4, 5, 5)$	$3\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5$
3	$(2, 3, 4, 5, 6)$	$3\Lambda_0 - 3\alpha_0 - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$
3	$(3, 3, 3, 3, 3)$	$3\Lambda_0 - 3\alpha_0 - 3\alpha_2 - 3\alpha_3 - 3\alpha_4 - 3\alpha_5$

Elements of $kC_{af} \cap (\overline{\Lambda} + \overline{Q})$

Theorem

Given $\mathring{A}$ the Cartan matrix for type B_n of finite type, the set of solutions to

$$\begin{cases} \mathring{A}\mathbf{x} \geq 0 \\ x_2 \leq k \end{cases}$$

is $\{\mathbf{x} = \mathbf{0}\} \cup \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n_{\geq 0} | 1 \leq x_2 \leq k, x_1 = \left\lceil \frac{x_2}{2} \right\rceil + l_1, 0 \leq l_1 \leq \left\lfloor \frac{x_2}{2} \right\rfloor, x_i = x_2 + \sum_{j=3}^{i} l_j, 0 \leq l_3 \leq \left\lfloor \frac{x_2}{2} \right\rfloor - l_1, 0 \leq l_n \leq l_{n-1} \leq l_{n-2} \leq \cdots \leq l_4 \leq l_3$ *for all* $2 \leq i \leq n\}$.

x_2	$\mathbf{x}$ vector	Element of $\max(4\Lambda_0) \cap P^+$
4	$(2,4,4,4,4)$	$4\Lambda_0 - 2\alpha_0$
4	$(2,4,5,5,5)$	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$
4	$(2,4,5,6,6)$	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3$
4	$(2,4,5,6,7)$	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
4	$(2,4,6,6,6)$	$4\Lambda_0 - 3\alpha_0 - \alpha_1 - 2\alpha_2$
4	$(2,4,6,7,7)$	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5$
4	$(2,4,6,7,8)$	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4$
4	$(2,4,6,8,8)$	$4\Lambda_0 - 4\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3$
4	$(2,4,6,8,9)$	$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4 - \alpha_5$
4	$(2,4,6,8,10)$	$4\Lambda_0 - 5\alpha_0 - 3\alpha_1 - 6\alpha_2 - 4\alpha_3 - 2\alpha_4$
4	$(3,4,4,4,4)$	$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5$
4	$(3,4,5,5,5)$	$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$
4	$(3,4,5,6,6)$	$4\Lambda_0 - 3\alpha_0 - 2\alpha_2 - \alpha_3$
4	$(3,4,5,6,7)$	$4\Lambda_0 - 4\alpha_0 - \alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$
4	$(4,4,4,4,4)$	$4\Lambda_0 - 4\alpha_2 - 4\alpha_3 - 4\alpha_4 - 4\alpha_5$

$\max k\Lambda_0 \cap P_+$

Theorem

Let $n \geq 3$, $\Lambda = k\Lambda_0, k \geq 2$. *Then* $\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (l - (\lceil \frac{x_2}{2} \rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - \left(\sum_{i=3}^{n} (2l - (x_2 + \sum_{j=3}^{i} l_j))\alpha_i \right)$ *where*

- $1 \leq x_2 \leq k$
- $l = \max\{x_1, \lceil \frac{x_n}{2} \rceil\}$
- $0 \leq l_1 \leq \lfloor \frac{x_2}{2} \rfloor$
- $0 \leq l_3 \leq \lfloor \frac{x_2}{2} \rfloor - l_1$
- $0 \leq l_n \leq l_{n-1} \leq \cdots \leq l_4 \leq l_3$
- $l_2 = x_2 - x_1$ *for* $n = 3$, $l_2 = 0$ *else*

Thank you!

Type $C_n^{(1)}$

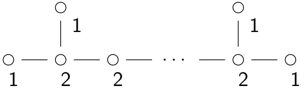
$$\underset{1}{\circ} \Rightarrow \underset{2}{\circ} — \cdots — \underset{2}{\circ} \Leftarrow \underset{1}{\circ}$$

Theorem

Let $n \geq 2$, $\Lambda = k\Lambda_0, k \geq 2, m \in \mathbb{Z}_{\geq 0}$. *Then*
$\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (2l - x_1)\alpha_1 - \left(\sum_{i=2}^{n-1} (2l - (x_1 + \sum_{j=2}^{i} l_j))\alpha_i \right)$
where

- $1 \leq x_1 \leq k$
- $0 \leq l_2 \leq x_1$
- $0 \leq l_{n-1} \leq l_{n-2} \leq \cdots \leq l_3 \leq l_2$ *and* $l_{n-1} = x_1$ *for* $n = 2$
- $\left\lceil \frac{x_{n-1}}{2} \right\rceil \leq l \leq \left\lfloor \frac{x_{n-1} + l_{n-1}}{2} \right\rfloor$

Type $D_n^{(1)}$



Theorem

Let $n \geq 4$, $\Lambda = k\Lambda_0, k \geq 2, m \in \mathbb{Z}_{\geq 0}$. *Then*
$\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (l - (\lceil \frac{x_2}{2} \rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - \left(\sum_{i=3}^{n-2}(2l - (x_2 + \sum_{j=3}^{i} l_j))\alpha_i\right) - (l - x_{n-1})\alpha_{n-1} - (l - x_n)\alpha_n$ *where*

- $2 \leq x_2 \leq k$
- $l = \max\{x_1, x_{n-1}, x_n\}$
- $0 \leq l_1 \leq \lfloor \frac{x_2}{2} \rfloor$
- $0 \leq l_3 \leq \lfloor \frac{x_2}{2} \rfloor - l_1$
- $0 \leq l_{n-2} \leq l_{n-3} \leq \cdots \leq l_4 \leq l_3$, $l_2 = x_2 - x_1$ *for* $n = 4$, $l_2 = 0$ *else*
- $x_{n-1} + x_n \leq x_{n-2} + l_{n-2}$
- $\min\{x_{n-1}, x_n\} \geq \lceil \frac{x_{n-2}}{2} \rceil$

Type $G_2^{(1)}$

$$\underset{1}{\circ} - \underset{2}{\circ} \Rrightarrow \underset{3}{\circ}$$

Theorem

Let $\Lambda = k\Lambda_0, k \geq 2$. *Then*
$\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2$ *where*

- $1 \leq x_1 \leq k$
- $\left\lceil \frac{3x_1}{2} \right\rceil \leq x_2 \leq 2x_1$
- $l = \left\lceil \frac{x_2}{3} \right\rceil$

Type $F_4^{(1)}$

$$\underset{1}{\circ} - \underset{2}{\circ} - \underset{3}{\circ} \Rightarrow \underset{4}{\circ} - \underset{2}{\circ}$$

Theorem

Let $\Lambda = k\Lambda_0, k \geq 2$. *Then* $\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - (x_1 + l_2))\alpha_2 - (4l - (x_1 + l_2 + l_3))\alpha_3 - (2l - x_4)\alpha_4$ *where*

- $1 \leq x_1 \leq k$
- $\left\lceil \frac{x_1}{2} \right\rceil \leq l_2 \leq x_1$
- $\left\lceil \frac{x_1 + l_2}{3} \right\rceil \leq l_3 \leq l_2$
- $\left\lceil \frac{x_1 + l_2 + l_3}{2} \right\rceil \leq x_4 \leq 2l_3$
- $l = \left\lceil \frac{x_4}{2} \right\rceil$

Type $A_{2n}^{(2)}$

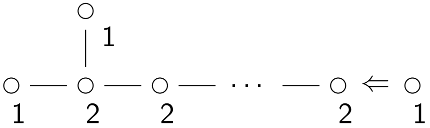
$$\begin{array}{ccccccccc} \bigcirc & \Leftarrow & \bigcirc & — & \cdots & — & \bigcirc & \Leftarrow & \bigcirc \\ 2 & & 2 & & & & 2 & & 1 \end{array}$$

Theorem

Let $n \geq 2$, $\Lambda = k\Lambda_0, k \geq 2, m \in \mathbb{Z}_{\geq 0}$. *Then*
$\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (l - x_1)\alpha_1 - \left(\sum_{i=2}^{n-1} (l - (x_1 + \sum_{j=2}^{i} l_j))\alpha_i \right)$
where

- $1 \leq x_1 \leq \left\lfloor \frac{k}{2} \right\rfloor$
- $0 \leq l_2 \leq x_1$ *for* $n > 2$
- $0 \leq l_{n-1} \leq l_{n-2} \leq \cdots \leq l_3 \leq l_2$ *and* $l_{n-1} = x_1$ *for* $n = 2$
- $\left\lceil \frac{x_{n-1}}{2} \right\rceil \leq l \leq \left\lfloor \frac{x_{n-1} + l_{n-1}}{2} \right\rfloor$

Type $A_{2n-1}^{(2)}$



Theorem

Let $n \geq 3$, $\Lambda = k\Lambda_0, k \geq 2, m \in \mathbb{Z}_{\geq 0}$. *Then* $\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (l - (\lceil \frac{x_2}{2} \rceil + l_1))\alpha_1 - (2l - x_2)\alpha_2 - \left(\sum_{i=3}^{n-1} (2l - (x_2 + \sum_{j=3}^{i} l_j))\alpha_i \right) - (l - x_n)\alpha_n$ *where*

- $2 \leq x_2 \leq k$
- $l = \max\{x_1, x_n\}$
- $0 \leq l_1 \leq \lfloor \frac{x_2}{2} \rfloor$
- $0 \leq l_3 \leq \lfloor \frac{x_2}{2} \rfloor - l_1$ *for* $n > 3$
- $0 \leq l_{n-1} \leq l_{n-2} \leq \cdots \leq l_4 \leq l_3$, $l_2 = x_2 - x_1$ *for* $n = 3$, $l_2 = 0$ *else*
- $\lceil \frac{x_{n-1}}{2} \rceil \leq x_n \leq \left\lfloor \frac{x_{n-1} + l_{n-1}}{2} \right\rfloor$

Type $D_{n+1}^{(2)}$

$$\underset{1}{\circ} \Leftarrow \underset{1}{\circ} — \cdots — \underset{1}{\circ} \Rightarrow \underset{1}{\circ}$$

Theorem

Let $n \geq 2$, $\Lambda = k\Lambda_0, k \geq 2, m \in \mathbb{Z}_{\geq 0}$. Then

$\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (l - x_1)\alpha_1 - \left(\sum_{i=2}^{n} (l - (x_1 + \sum_{j=2}^{i} l_j))\alpha_i \right)$

where

- $1 \leq x_1 \leq \left\lfloor \frac{k}{2} \right\rfloor$
- $l = x_n$
- $0 \leq l_2 \leq x_1$
- $0 \leq l_n \leq l_{n-1} \leq \cdots \leq l_4 \leq l_3$

Type $E_6^{(2)}$

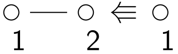
$$\begin{array}{ccccccccc} \circ & - & \circ & - & \circ & \Leftarrow & \circ & - & \circ \\ 1 & & 2 & & 3 & & 2 & & 1 \end{array}$$

Theorem

Let $\Lambda = k\Lambda_0, k \geq 2$. *Then*
$\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (2l - x_1)\alpha_1 - (3l - x_2)\alpha_2 - (2l - x_3)\alpha_3$
where

- $2 \leq x_1 \leq k$
- $3\left\lceil \frac{x_1}{2} \right\rceil \leq x_2 \leq 2x_1$
- $\left\lceil \frac{2}{3}x_2 \right\rceil \leq x_3 \leq x_2 - \left\lceil \frac{x_1}{2} \right\rceil$
- $\left\lceil \frac{x_3}{2} \right\rceil \leq l \leq 2x_3 - x_2$

Type $D_4^{(3)}$



$$\begin{cases} 2x_1 - 3x_2 & \geq 0 \\ -x_1 + 2x_2 & \geq 0 \\ x_1 & \leq k \end{cases}$$

Elements of $kC_{af} \cap (\overline{\Lambda} + \overline{Q})$

Theorem

The set of solutions to

$$\begin{cases} 2x_1 - 3x_2 & \geq 0 \\ -x_1 + 2x_2 & \geq 0 \\ x_1 & \leq k \end{cases}$$

is $\{\mathbf{x} = \mathbf{0}\} \cup \{\mathbf{x} = (x_1, x_2) | 2 \leq x_1 \leq k, \left\lceil \frac{x_1}{2} \right\rceil \leq x_2 \leq \left\lfloor \frac{2x_1}{3} \right\rfloor \}$.

Proof.

Note that since A is of finite type, $\mathbf{x} \geq 0$ and so $x_1 \geq 0$ and $x_1 \leq k$ by definition. Therefore, $x_1 = 0$, which implies $\mathbf{x} = 0$, or $x_1 \geq 1$. However, by $(A\mathbf{x})_1 \geq 0$ and $(A\mathbf{x})_2 \geq 0$, $\frac{x_1}{2} \leq x_2 \leq \frac{2}{3}x_1$ and since x_2 must be an integer, $x_1 \neq 1$.
Now, fix x_1 such that $2 \leq x_1 \leq k$. Then by $(A\mathbf{x})_1 \geq 0$, $x_2 \leq \frac{2}{3}x_1$ and by $(A\mathbf{x})_2 \geq 0, x_2 \geq \frac{1}{2}x_1$. Since x_2 must be an integer, we have $\left\lceil \frac{x_1}{2} \right\rceil \leq x_2 \leq \left\lfloor \frac{2x_1}{3} \right\rfloor$.
This describes all possible solutions. □

$\max k\Lambda_0 \cap P_+$

Theorem

Let $\Lambda = k\Lambda_0, k \geq 2$. *Then* $\max(\Lambda) \cap P^+ = \{\Lambda\} \cup \{\Lambda - l\alpha_0 - (2l - x_1)\alpha_1$ *where*

- $2 \leq x_1 \leq k$
- $\left\lceil \frac{x_1}{2} \right\rceil \leq l \leq \left\lfloor \frac{2x_1}{3} \right\rfloor$