Primitive ideals and finite W-algebras of low rank

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June 4, 2018

Primitive ideals and associated varieties

- $\mathfrak{g}$ is a semisimple finite dimensional complex Lie algebra.
- Question: What are the simple $\mathfrak{g}$-modules?
- Easier question: What are the primitive ideals of $U(\mathfrak{g})$?
  (An ideal in some algebra is *primitive* if it is the annihilator of a simple module).
- Duflo: Every primitive ideal in $U(\mathfrak{g})$ is equal to the annihilator of a simple highest weight module.
- Joseph: If I is a primitive ideal $U(\mathfrak{g})$ then the associated variety $\mathcal{VA}(I) = \overline{G \cdot e}$, the closure of a nilpotent orbit.
  ($\mathcal{VA}(I) = Z(I)$, when we consider I to be contained in $S(\mathfrak{g}^*) = \mathbb{C}[\mathfrak{g}]$.)
- In summary:
  $\{\text{simple } \mathfrak{g}\text{-modules}\} \twoheadrightarrow \text{Prim } U(\mathfrak{g}) \twoheadrightarrow \{\text{nilpotent orbits}\}$.
- Finite W-algebras fit in quite nicely with this picture.

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Finite W-algebras

- A finite W-algebra is denoted $U(\mathfrak{g}, e)$ where e is a nilpotent element of $\mathfrak{g}$.

Extreme cases: $U(\mathfrak{g}, 0) = U(\mathfrak{g})$
$U(\mathfrak{g}, e_{\text{reg}}) \cong Z(\mathfrak{g})$ (Kostant)

- In general we can think of $U(\mathfrak{g}, e)$ as living somewhere between $Z(\mathfrak{g})$ and $U(\mathfrak{g})$.
- $Z(\mathfrak{g})$ embeds into $U(\mathfrak{g}, e)$ and the center of $U(\mathfrak{g}, e)$ is $Z(\mathfrak{g})$.
- $U(\mathfrak{g}, e)$ is a deformation of $U(\mathfrak{g}^e)$ (and also of $S(\mathfrak{g}^e)$), where $\mathfrak{g}^e = \{x \in \mathfrak{g} \mid [x, e] = 0\}$.

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Definition of finite W algebra $U(\mathfrak{g}, e)$

- Start with nilpotent $e \in \mathfrak{g}$. By Jacobson-Morozov Theorem, e embeds in to $\mathfrak{sl}_2$-triple (e, h, f).
- Let $(\cdot, \cdot)$ denote a non-degenerate equivariant symmetric bilinear form on $\mathfrak{g}$.
- $\mathfrak{sl}_2$ representation theory implies that $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$, where $\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}$.
- Define $\chi : \mathfrak{g}(\leq -1) \to \mathbb{C}$ via $\chi(m) = (e, m)$.
- Let $\mathfrak{l}$ be a maximal isotropic subspace of $\mathfrak{g}(-1)$ under the form $\langle x, y \rangle = \chi([x, y])$, and let $\mathfrak{l}^\perp$ be the complementary maximal isotropic subspace
- Let $\mathfrak{m} = \mathfrak{g}(\geq 0) \oplus \mathfrak{l}$, let $\mathfrak{n} = \mathfrak{g}(< -1) \oplus \mathfrak{l}^\perp$.
- Let I be the left ideal of $U(\mathfrak{g})$ generated by $\{m - \chi(m) \mid m \in \mathfrak{m}\}$.
- $U(\mathfrak{g}, e) = (U(\mathfrak{g})/I)^{\mathfrak{n}} = \{u + I \in U(\mathfrak{g})/I \mid [\mathfrak{n}, u] \subseteq I\}$

Representation theory of finite W-algebras

- $U(\mathfrak{g})/I$ is a $(U(\mathfrak{g}), U(\mathfrak{g}, e))$-bimodule, so there exists a functor $S : U(\mathfrak{g}, e)\text{-mod} \to U(\mathfrak{g})\text{-mod}$, $V \mapsto U(\mathfrak{g})/I \otimes_{U(\mathfrak{g},e)} V$. A result of Skryabin says that S is a categorical equivalence between $U(\mathfrak{g}, e)$-mod and *Whittaker modules for e*, ie modules on which $m - \chi(m)$ acts locally nilpotently for all $m \in \mathfrak{m}$.
- Losev has defined a map

$$\cdot^\dagger : \text{Prim}\, U(\mathfrak{g}, e) \to \text{Prim}\, U(\mathfrak{g}).$$

- This map restricts to a surjection:

$$\cdot^\dagger : \text{Prim}_{\text{fd}}\, U(\mathfrak{g}, e) \twoheadrightarrow \text{Prim}_{\overline{G \cdot e}}\, U(\mathfrak{g}).$$

- The fibers of this map are Γ-orbits, where $\Gamma = G^e/(G^e)^\circ$.
- This suggests that the finite dimensional representation theory of $U(\mathfrak{g}, e)$ should be able to tell us something about the infinite dimensional representation theory of $U(\mathfrak{g})$.

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Induction of ideals and orbits

- Ideals in $U(\mathfrak{g})$ and nilpotent orbits can be induced from Levi subalgebras of $\mathfrak{g}$.
- Let $\mathfrak{g}'$ be a Levi subalgebra of $\mathfrak{g}$. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ such that $\mathfrak{p} = \mathfrak{g}' \oplus \mathfrak{u}$ for some nilpotent subalgebra $\mathfrak{u}$.
- If $\mathcal{O}'$ is a nilpotent orbit in $\mathfrak{g}'$, then the $\mathcal{O}$ is the orbit induced from $\mathcal{O}'$ is the unique nilpotent orbit in $\mathfrak{g}$ such that $\mathcal{O} \cap (\mathcal{O}' + \mathfrak{u})$ is open in $\mathcal{O}' + \mathfrak{u}$.
- If I' is an ideal of $\mathfrak{g}'$, then we define the induced ideal $\mathcal{I}_{\mathfrak{g}'}^{\mathfrak{g}}(I')$ to be the largest two-sided ideal of $U(\mathfrak{g})$ which contained in the left ideal $U(\mathfrak{g})(\mathfrak{u} + I')$.
- If $I' = \mathrm{Ann}_{U(\mathfrak{g}')}(M')$ for some simple $\mathfrak{g}'$-module M', then $\mathcal{I}_{\mathfrak{g}'}^{\mathfrak{g}}(I') = \mathrm{Ann}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M')$
- Even if I' is primitive, $\mathcal{I}_{\mathfrak{g}'}^{\mathfrak{g}}(I')$ need not be.
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Completely prime and multiplicity-free primitive ideals

- An ideal I in $U(\mathfrak{g})$ is *completely prime* if $U(\mathfrak{g})/I$ has no zero-divisors.
- The classification of completely prime primitive ideals of $U(\mathfrak{g})$ is still an open problem outside of type A.
- Outside of type A, not every completely prime primitive ideal can be induced from a proper Levi subalgebra.
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Completely prime primitive ideals and 1-dimensional $U(\mathfrak{g}, e)$-modules

- Let $\mathcal{E}$ denote the variety of 1-dimensional $U(\mathfrak{g}, e)$-modules, and let $\mathcal{E}^\Gamma$ denote Γ-invariant elements of $\mathcal{E}$.
- Premet has shown that under Skryabin's equivalence, elements of $\mathcal{E}$ correspond to completely prime primitive ideals in $U(\mathfrak{g})$.
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Theorem (Premet, Topley)

Let $\mathfrak{g}$ be any semi-simple Lie algebra, and let $e \in \mathfrak{g}$ be an induced nilpotent element. Let $n = \dim(\mathfrak{g}^e/[\mathfrak{g}^e, \mathfrak{g}^e])^\Gamma$. If $(\mathfrak{g}, G \cdot e)$ is not one of the six cases listed below, then $\mathcal{E}^\Gamma \cong \mathbb{C}^n$.

Corollary (Premet, Topley)

If $I \subseteq U(\mathfrak{g})$ is a multiplicity free primitive ideal whose associated variety is induced from nilpotent orbit in a proper Levi subalgebra and is not one of the six cases listed below, then I is induced from an ideal in a proper Levi subalgebra.

- Bala-Carter labels of unresolved cases: (F_4,$C_3(a_1)$), (E_6, A_3+A_1), (E_7, $D_6(a_2)$), (E_8,$E_6(a_3)$ + A_1), (E_8, $D_6(a_2)$), (E_8, $E_7(a_2)$), (E_8, $E_7(a_5)$).

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Our contribution

- Except for the 6 cases listed above, Premet and Topley calculated $\mathcal{E}^{\Gamma}$ for all induced nilpotent orbits in semi-simple finite dimensional Lie algebras.
- Our goal: Use computers to calculate $\mathcal{E}$, as well as $\mathcal{E}^{\Gamma}$ in the cases Premet and Topley did not do.
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Results

Type	Orbit	Γ	$\mathcal{E}$	$\mathcal{E}^\Gamma$
C_2	(2,2)	S_2	$\mathbb{C} \sqcup \mathbb{C}/\mathbb{C}^0$	$\mathbb{C}$
C_3	(4,2)	S_2	$\mathbb{C}^2 \sqcup \mathbb{C}^2/\mathbb{C}$	$\mathbb{C}^2$
B_3	(5,1,1)	S_2	$\mathbb{C}^2 \sqcup \mathbb{C}^2/\mathbb{C}$	$\mathbb{C}^2$
C_4	(4,2,2)	S_2	$\mathbb{C}^2 \sqcup \mathbb{C}/\mathbb{C}^0$	$\mathbb{C}^2$
C_4	(6,2)	S_2	$\mathbb{C}^3 \sqcup \mathbb{C}^3/\mathbb{C}^2$	$\mathbb{C}^3$
B_4	(7,1,1)	S_2	$\mathbb{C}^3 \sqcup \mathbb{C}^3/\mathbb{C}^2$	$\mathbb{C}^3$
B_4	(5,3,1)	$S_2 \times S_2$	Three $\mathbb{C}^2$'s which intersect at a point and pairwise intersect at lines	$\mathbb{C}^2$
D_4	(3,3,1,1)	S_2	$\mathbb{C}^2 \sqcup \mathbb{C}/\mathbb{C}^0$	$\mathbb{C}$
G_2	$G_2(a_1)$	S_3	$\mathbb{C} \sqcup \mathbb{C} \sqcup \mathbb{C} \sqcup \mathbb{C}/\mathbb{C}^0$	$\mathbb{C}$
F_4	$C_3(a_1)$	S_2	$\mathbb{C} \sqcup \mathbb{C}^0 \sqcup \mathbb{C}^0 \sqcup \mathbb{C}^0$	$\mathbb{C} \sqcup \mathbb{C}^0$
F_4	$F_4(a_1)$	S_2	$\mathbb{C}^3 \sqcup \mathbb{C}^3/\mathbb{C}^2$	$\mathbb{C}^3$
F_4	$F_4(a_2)$	S_2	$\mathbb{C}^2 \sqcup \mathbb{C}^2/\mathbb{C}$	$\mathbb{C}^2$
F_4	$F_4(a_3)$	S_4	$\left(\left(\bigsqcup_{i=1}^5 \mathbb{C}\right) \sqcup \left(\bigsqcup_{i=1}^3 \mathbb{C}^2\right)\right)/\mathbb{C}^0$	$\mathbb{C}$
E_6	$A_3 + A_1$	1	$\mathbb{C} \sqcup \mathbb{C}^0$	$\mathbb{C} \sqcup \mathbb{C}^0$
E_6	$E_6(a_3)$	S_2	$\mathbb{C}^4 \sqcup \mathbb{C}^3/\mathbb{C}^2$	$\mathbb{C}^3$
E_6	$D_4(a_1)$	S_3	$\mathbb{C}^2 \sqcup \mathbb{C}^2 \sqcup \mathbb{C}^2 \sqcup \mathbb{C}^2 \sqcup \mathbb{C}/\mathbb{C}^0$	$\mathbb{C}$

Results

Theorem (B., Goodwin)

Let $\mathfrak{g}$ be of type F_4 and $\mathcal{O}$ with Bala–Carter label $\mathrm{C}_3(a_1)$, or let $\mathfrak{g}$ be of type E_6 and $\mathcal{O}$ with Bala–Carter label $\mathrm{A}_3 + \mathrm{A}_1$. Then there is a multiplicity free primitive ideal of $U(\mathfrak{g})$ with associated variety $\overline{\mathcal{O}}$ that cannot be induced from a primitive ideal of $U(\mathfrak{g}')$ for any proper Levi subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$.