Pieri rules and q-characters of Hernandez-Leclerc modules for quantum affine $\mathfrak{sl}_{n+1}$

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June 7, 2018

Overview

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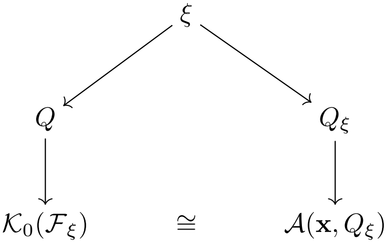
In the case when ξ is alternating ($\xi(i+1) = \xi(i) + 1 = \xi(i-1)$) or the monotonic case ($\xi(i+1) = \xi(i) + 1$) Hernandez and Leclerc proved that one has the following picture



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Our methods allow us to give the image of an arbitrary cluster variable in $\mathcal{K}_0(\mathcal{F}_\xi)$; equivalently we classify all the prime representations in $\mathcal{F}_\xi$. They turn out to be a generalization of minimal affinizations, and can be called "minimal by parts". These kind of prime representations had been studied in [CMY].

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We also develop a recursive formula for the cluster variables which translates to giving a q–character formula for the corresponding irreducible (HL) module in terms of the fundamental modules (the initial seed) and Kirillov–Reshetikhin modules (the frozen variables).

Notation

- $\mathbf{U}_q$: the quantized universal enveloping algebra of $\tilde{\mathfrak{sl}}_{n+1}$ (q not root of unity)
- Let $\mathcal{P}_\xi^+$ be the free monoid generated by the set $\left\langle \boldsymbol{\omega}_{i,q^{\xi(i)\pm 1}} : i \in I \right\rangle$.
- To each element $\boldsymbol{\pi}$ of $\mathcal{P}_\xi^+$ one can associate a (unique up to isomorphism) irreducible finite–dimensional module $V(\boldsymbol{\pi})$ of $\mathbf{U}_q$.
- Let $\mathcal{F}_\xi$ be the full subcategory of finite–dimensional representations whose irreducible constituents are the modules $V(\boldsymbol{\pi})$, $\boldsymbol{\pi} \in \mathcal{P}_\xi^+$.

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$$\boldsymbol{\omega}(i,j) := \prod_{k \in [i,j]_>} \boldsymbol{\omega}_{k,q^{\xi(k)-1}} \prod_{k \in [i,j]_<} \boldsymbol{\omega}_{k,q^{\xi(k)+1}},$$

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$$\begin{array}{cccccccccc} \xi(i): & 1 & & 2 & & 1 & & 2 & & 3 \\ Q: & 1 & \to & 2 & \leftarrow & 3 & \to & 4 & \to & 5 \end{array}$$

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$$\boldsymbol{\omega}(1,4) = \boldsymbol{\omega}_{1,1}\boldsymbol{\omega}_{2,q^3}\boldsymbol{\omega}_{3,1}\boldsymbol{\omega}_{4,q^3}, \quad \boldsymbol{\omega}(1,5) = \boldsymbol{\omega}_{1,1}\boldsymbol{\omega}_{2,q^3}\boldsymbol{\omega}_{3,1}\boldsymbol{\omega}_{5,q^4}.$$

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We shall call the modules associated to

$$\boldsymbol{\omega}_{i,q^{\xi(i)\pm 1}}, \quad \boldsymbol{\omega}(i,j), \quad \mathbf{f}_i = \boldsymbol{\omega}_{i,q^{\xi(i)+1}}\boldsymbol{\omega}_{i,q^{\xi(i)-1}},$$

HL-modules of type ξ.

Main Results

Theorem

Let $\mathbf{x} = (x_1, \cdots, x_n, f_1, \cdots, f_n)$. For any height function ξ the map $\iota : \mathcal{A}(\mathbf{x}, Q_\xi) \to \mathcal{K}_0(\mathcal{F}_\xi)$ given by

$$\iota(x_i) = \begin{cases} [\boldsymbol{\omega}_{i,q^{\xi(i)-1}}], & \xi(i) = \xi(i+1)+1, \\ [\boldsymbol{\omega}_{i,q^{\xi(i)+1}}], & \xi(i) = \xi(i+1)-1, \end{cases},$$

$$\iota(f_i) = [\mathbf{f}_i],$$

is an isomorphism of algebras. Further, the image of a an arbitrary cluster variable corresponds to a prime HL–module,

$$\iota(x[\alpha_{i,j}]) = \begin{cases} [\boldsymbol{\omega}(i, j+1)], & j \neq i_{p+1}, \\ [\boldsymbol{\omega}(i, i_p+1)], & i \leq i_p < j = i_{p+1}, \\ [\boldsymbol{\omega}_{i,a_i}], & i_p < i < j = i_{p+1}. \end{cases}$$

Suppose that $\alpha, \beta \in \Phi_{\geq -1}$ are such that $\iota(x[\alpha]) \otimes \iota(x[\beta])$ is reducible. Then $x[\alpha]x[\beta]$ is not a cluster monomial.

Corollary

The map ι maps cluster monomials to irreducible modules.

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Example

- $[\boldsymbol{\omega}(1,4)][\boldsymbol{\omega}_{3,q^2}] = [\boldsymbol{\omega}(1,2)][\mathbf{f}_3][\boldsymbol{\omega}_{4,q^3}] + [\boldsymbol{\omega}_{1,1}][\mathbf{f}_2][\mathbf{f}_4]$
- $[\boldsymbol{\omega}(1,4)][\boldsymbol{\omega}(3,5)] = [\boldsymbol{\omega}(1,5)][\boldsymbol{\omega}(3,4)] + [\mathbf{f}_1][\mathbf{f}_3][\mathbf{f}_5]$

$$\iota(x[\alpha_{1,3}]) = [\boldsymbol{\omega}(1,4)] = [\boldsymbol{\omega}_{1,1}\boldsymbol{\omega}_{2,q^3}\boldsymbol{\omega}_{3,1}\boldsymbol{\omega}_{4,q^3}]$$
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q-character formulae

Our next result gives an explicit formula for the character of the prime representations (cluster variables) in terms of fundamental representations and the KR-modules (the initial seed). For ease of notation we only state the result here in the case when ξ is alternating. Recall that in this case

$$\boldsymbol{\omega}(i,k) = \boldsymbol{\omega}_{i,\xi(i)}\boldsymbol{\omega}_{i+1,\xi(i+1)+2}\cdots .$$

Then, for $i<k$ we have

$$\iota^{-1}([\boldsymbol{\omega}(i,k)]) = x[\alpha_{i,k}] = \sum_{\mathbf{r}=(r_i,\cdots,r_k)} f_i^{r_i}\cdots f_k^{r_k} q_{i,k}^{\mathbf{r}},$$

where $r_j\in\{0,1\}$ and $r_j=0 \implies r_{j-1}=1=r_{j+1}$ and

$$q_{i,k}^{\mathbf{r}} = x_{i-1}^{1-r_i}x_{k+1}^{1-r_k}\prod_{j=i}^{k} x_j^{1-r_{j-1}-r_{j+1}},$$

where we understand $r_{i-1}=r_{k+1}=1$.

Methods

In $\mathcal{A}(\mathbf{x}, Q_\xi)$

- Describe a pattern for the a sequence of mutations of Q_ξ;
- Obtain a mutation formula for all cluster variables;

In $\mathcal{K}_0(\mathcal{F}_\xi)$

- Pieri rules: explicit decomposition of the tensor product of HL-modules with fundamental modules in $\mathcal{F}_\xi$.

Connection

- Compare mutation formula with Pieri rule.

The quiver Q_ξ

For the height function ξ we define a new quiver Q_ξ, with $(Q_\xi)_0 = I \cup I'$ and for $(Q_\xi)_1$ the arrows at $j \in I$ are as follows:

- If $\xi(j) = \xi(j+1) + 1 = \xi(j+2)$ then

$$\begin{array}{ccccc} j-1 & \longleftarrow & j & \longrightarrow & (j+1) \\ & & \uparrow & & \\ & & j' & & \end{array}$$

- If $\xi(j) = \xi(j+1) + 1 = \xi(j+2) + 2$ then

$$\begin{array}{ccccc} j-1 & \longleftarrow & j & \longleftarrow & (j+1) \\ & & \uparrow & \searrow & \\ & & j' & & (j+1)' \end{array}$$

Example

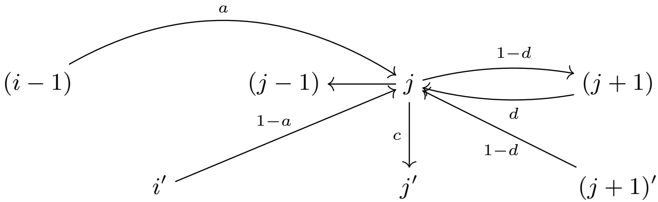
For ξ as before we have Q_ξ given by

$$\begin{array}{ccccccccc} 1 & \longleftarrow & 2 & \longrightarrow & 3 & \longrightarrow & 4 & \longleftarrow & 5 \\ \downarrow & & \uparrow & & \downarrow & \nwarrow & \downarrow & & \uparrow \\ 1' & & 2' & & 3' & & 4' & & 5' \end{array}$$

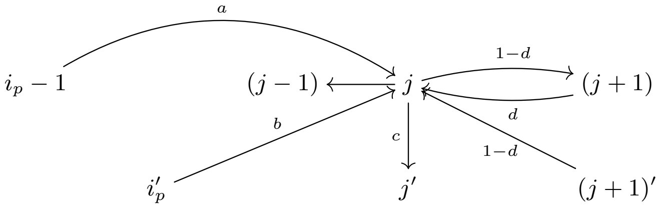
Suppose that $i < j$ and that we have an arrow $(j-1) \to j$ in Q_ξ. Let $1 = i_1 < i_2 < \cdots < i_k$ be the set of sinks and sources of Q_ξ. Then in $Q_\xi[i, j-1]$ the edges at the node j are as follows: Set

$$a = 1 - \delta_{i,i_p}, \quad c = \delta_{j,i_p+1}, \quad b = \delta_{i_{p-1}+1,i_p}, \quad d = \delta_{j,i_{p+1}}$$

- if $i_p \leq i < j \leq i_{p+1}$, then



- if $i < i_p \leq j \leq i_{p+1}$, then



Cluster Algebras

The non–frozen cluster variables $x[\alpha]$, $\alpha \in R^+ \cup \{-\alpha_i : 1 \leq i \leq n\}$ are given as follows. We adopt the convention that $\alpha_{i,k} = \alpha_k$ if $k \leq i$ and $\alpha_{i,k} = \alpha_i + \cdots + \alpha_k$ if $k > i$. Let $1 = i_1 < \cdots < i_k = n$, be the sinks and sources of Q_ξ

$$x[-\alpha_i] = x_i,$$
$$x_i x[\alpha_i] = (1 - \delta_{\xi(i),\xi(i+2)})\,(f_i x_{i+1} + f_{i+1} x_{i-1}) + \delta_{\xi(i),\xi(i+2)}(f_i + x_{i-1} x_{i+1}).$$

1. if $i_p \leq i < j \leq i_{p+1}$. Then,

$$x_j x[\alpha_{i,j}] = f_j^c x[\alpha_{i,j-1}] x_{j+1} + f_{j+1} f_i^{1-a} x_{i-1}^a, \quad \text{if } j < i_{p+1}$$
$$x_j x[\alpha_{i,j}] = f_i^{1-a} x_{j+1} x_{i-1}^a + f_j^c x[\alpha_{i,j-1}, \quad \text{if } j = i_{p+1}.$$

2. if $i < i_p < j \leq i_{p+1}$.

$$x_j x[\alpha_{i,j}] = f_j^c x[\alpha_{i,j-1}] x_{j+1} + f_{j+1} f_{i_p}^b x[\alpha_{i,i_p-1}], \quad \text{if } j < i_{p+1}$$
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