Classification of integrable modules of twisted full toroidal Lie algebras

Punita Batra

Harish-Chandra Research Institute
Allahabad, INDIA

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Preliminaries

- All vector spaces, algebras and tensor products are over complex numbers $\mathbb{C}$. Let $\mathbb{Z}, \mathbb{N}$ and $\mathbb{Z}_+$ denote integers, non-negative integers and positive integers.
- Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra and let $(,)$ be a non-degenerate symmetric bilinear form on $\mathfrak{g}$. We fix a positive integer n. Let $\sigma_0, \sigma_1, \cdots, \sigma_n$ be commuting finite order automorphisms of $\mathfrak{g}$ of order $m_0, m_1, \cdots, m_n$ respectively. Let $m = (m_1, \cdots, m_n) \in \mathbb{Z}^n$. Let $k = (k_1, \cdots, k_n)$ and $l = (l_1, \cdots, l_n)$ denote vectors in $\mathbb{Z}^n$.

- Let $\Gamma = m_1\mathbb{Z} \oplus \cdots \oplus m_n\mathbb{Z}$ and $\Gamma_0 = m_0\mathbb{Z}$. Let $\Lambda = \mathbb{Z}^n/\Gamma$ and $\Lambda_0 = \mathbb{Z}/\Gamma_0$. Let $\overline{k}$ and $\bar{l}$ denote the images in Λ. For any integers k_0 and l_0, let $\overline{k}_0$ and $\bar{l}_0$ denote the images in Λ_0. Let

$$\begin{aligned}
A &= \mathbb{C}[t_0^{\pm 1}, \cdots, t_n^{\pm 1}], \\
A_n &= \mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}], \\
A(m) &= \mathbb{C}[t_1^{\pm m_1}, \cdots, t_n^{\pm m_n}], \\
A(m_0, m) &= \mathbb{C}[t_0^{\pm m_0}, t_1^{\pm m_1}, \cdots, t_n^{\pm m_n}].
\end{aligned}$$

For $k \in \mathbb{Z}^n$, let $t^k = t_1^{k_1} \cdots t_n^{k_n} \in A_n$. Let ΩA be the vector space spanned by symbols $t_0^{k_0} t^k K_i, 0 \leq i \leq n, k_0 \in \mathbb{Z}, k \in \mathbb{Z}^n$. Let dA be the subspace spanned by $\sum_{i=0}^{n} k_i t_0^{k_0} t^k K_i$. Let $L(\mathfrak{g}) = \mathfrak{g} \otimes A$ and define toroidal Lie algebra

$$\widetilde{L}(\mathfrak{g}) = L(\mathfrak{g}) \oplus \Omega A / dA.$$

Let $X(k_0, k) = X \otimes t_0^{k_0} t^k$ and $Y = Y \otimes t_0^{l_0} t^l$ for $X, Y \in \mathfrak{g}, k_0, l_0 \in \mathbb{Z}$ and $k, l \in \mathbb{Z}^n$.
$[X(k_0, k), Y(l_0, l)] = [X, Y](l_0 + k_0, l + k) + (X, Y) \sum k_i t_0^{l_0 + k_0} t^{l+k} K_i$.
$\Omega A / dA$ is central.

We will now define multiloop algebra as a subalgebra of $L(\mathfrak{g})$. For
$0 \leq i \leq n$, let ξ_i be a m_ith primitive root of unity.
Let

$$\mathfrak{g}(\overline{k}_0, \overline{k}) = \{x \in \mathfrak{g} | \sigma_i x = \xi_i^{k_i} x, 0 \leq i \leq n\}.$$

Then define

$$L(\mathfrak{g}, \sigma) = \bigoplus_{(k_o, k) \in \mathbb{Z}^{n+1}} \mathfrak{g}(\overline{k}_0, \overline{k}) \otimes t_0^{k_0} t^k,$$

which is called a multiloop algebra. The finite dimensional irreducible modules for $L(\mathfrak{g}, \sigma)$ are classified by Michael Lau.

- We will now define the universal central extension of $L(\mathfrak{g},\sigma)$. Define $\Omega A(m_0,m)$ and $dA(m_0,m)$ similar to the definition of ΩA and dA by replacing A by $A(m_0,m)$. Denote $Z(m_0,m)=\Omega A(m_0,m)/dA(m_0,m)$ and note that $Z(m_0,m)\subseteq \Omega A/dA$.
  Define
  $$\widetilde{L}(\mathfrak{g},\sigma)=L(\mathfrak{g},\sigma)\oplus Z(m_0,m).$$
  Let $X\in\mathfrak{g}(\overline{k}_0,\overline{k})$ and $Y\in\mathfrak{g}(\overline{l}_0,\overline{l})$ and let $X(k_0,k)=X\otimes t_0^{k_0}t^k$ and $Y(l_0,l)=Y\otimes t_0^{l_0}t^l$. Define

(a) $[X(k_0,k),Y(l_0,l)]=[X,Y](k_0+l_0,k+l)+(X,Y)\sum k_i t_0^{l_0+k_0}t^{l+k}K_i$.

(b) $Z(m_0,m)$ is central.

Derivation algebra of $A(m_0, m)$ and its extension to $Z(m_0, m)$.

- Let $D(m_0, m)$ be the derivation algebra of $A(m_0, m)$. From now onwards we let s and r to be in Γ and s_0 and r_0 to be in Γ_0.
  For $0 \leq i \leq n$, consider $t_0^{s_0} t^s t_i \frac{d}{dt_i}$
  which acts on $A(m_0, m)$ as
  derivations. It is well known that $D(m_0, m)$ has the following basis

$$\{t_0^{s_0} t^s t_i \frac{d}{dt_i} | 0 \leq i \leq n, s_o \in \Gamma_0, s \in \Gamma\}.$$

  Let $d_i = t_i \frac{d}{dt_i}$ and it is easy to see that
- $[t_0^{s_0} t^s d_a, t_0^{r_0} t^r d_b] = r_a t_0^{r_0+s_0} t^{r+s} d_b - s_b t_0^{r_0+s_0} t^{r+s} d_a.$

$D(m_0, m)$ acts on $Z(m_0, m)$ in the following way

$t_0^{s_0} t^s d_a.(t_0^{r_0} t^r K_b) = r_a t_0^{r_0+s_0} t^{r+s} K_b + \delta_{ab} \sum_{p=0}^{n} s_p t_0^{r_0+s_0} t^{r+s} K_p$. It is known that $D(m_0, m)$ admits two non-trivial 2-cocycles with values in $Z(m_0, m)$.

$$\begin{aligned} \varphi_1(t_0^{r_0} t^r d_a, t_0^{s_0} t^s d_b) &= -s_a r_b \sum_{p=0}^{n} r_p t_0^{r_0+s_0} t^{r+s} K_p, \\ \varphi_2(t_0^{r_0} t^r d_a, t_0^{s_0} t^s d_b) &= r_a s_b \sum_{p=0}^{n} r_p t_0^{r_0+s_0} t^{r+s} K_p. \end{aligned}$$

Let φ be arbitrary linear combinations of φ_1 and φ_2. Then there is a corresponding Lie algebra $\tau = L(\mathfrak{g}, \sigma) \oplus Z(m_0, m) \oplus D(m_0, m)$.
The Lie brackets are defined in the following way.

$[t_0^{r_0} t^r d_a, X(k_0, k)] = k_a X(k_0 + r_0, k + r)$,

$[t_0^{r_0} t^r d_a, t^{s_0} t^s K_b] = s_a t_0^{r_0+s_0} t^{r+s} K_b + \delta_{ab} \sum_{p=0}^{n} r_p t_0^{r_0+s_0} t^{r+s} K_p$,

$[t_0^{r_0} t^r d_a, t_0^{s_0} t^s d_b] = s_a t_0^{r_0+s_0} t^{r+s} d_b - r_b t_0^{r_0+s_0} t^{r+s} d_a + \varphi(t_0^{r_0} t^r d_a, t^{s_0} t^s d_b)$,
where $r, s \in \Gamma, r_0, s_0 \in \Gamma_0, X \in \mathfrak{g}(\overline{k}_0, \overline{k})$.

Assumptions

(a) $\mathfrak{g}(\bar{0},\bar{0})$ is simple Lie algebra.

(b) We can choose Cartan subalgebra $\mathfrak{h}(0)$ and $\mathfrak{h}$ for $\mathfrak{g}(\bar{0},\bar{0})$ and $\mathfrak{g}$ such that $\mathfrak{h}(0) \subseteq \mathfrak{h}$.

(c) It is known that $\Delta_0^\times = \Delta(\mathfrak{g}(\bar{0},\bar{0}), \mathfrak{h}(0))\backslash\{0\}$ is an irreducible reduced finite root system and has atmost two root lengths. Let $\Delta_{0,sh}^\times$ be the set of non-zero short roots. Define

$$\Delta_{0,en}^\times = \begin{cases} \Delta_0^\times \cup 2\Delta_{0,sh}^\times \text{ if } \Delta_0^\times \text{ is of type } B_l \\ \Delta_0^\times \quad \text{otherwise} \end{cases} \qquad \Delta_{0,en} = \Delta_{0,en}^\times \cup \{0\}.$$

We assume that $\Delta(\mathfrak{g}, \mathfrak{h}(0)) = \Delta_{0,en}$.

Root space decomposition and integrable modules for τ.

- First note the center of τ is spanned by $K_0, K_1, \cdots, K_n$. Let $H = \mathfrak{h}(0) \oplus \sum \mathbb{C}K_i \oplus \sum \mathbb{C}d_i$ which is an abelian Lie algebra of τ and plays the role of Cartan subalgebra. Define $\delta_i, w_i \in H^*(0 \leq i \leq n)$ be such that
-
$$\begin{aligned} w_i(\mathfrak{h}(0)) &= 0, w_i(K_j) = \delta_{ij}, w_i(d_j) = 0, \\ \delta_i(\mathfrak{h}(0)) &= 0, \delta_i(K_j) = 0, \delta_i(d_j) = \delta_{ij}. \end{aligned}$$

Let $\delta_k = \sum_{i=1}^{n} k_i\delta_i$ *for* $k \in \mathbb{Z}^n$.

- Let $\mathfrak{g}(\overline{k}_0, \overline{k}, \alpha) = \{x \in \mathfrak{g}(\overline{k}_0, \overline{k}) | [h, x] = \alpha(h)x$ for all $h \in \mathfrak{h}(0)\}$ then τ has a root space decomposition. $\tau = \bigoplus_{\beta\in\Delta} \tau_\beta$ where

  $\Delta \subseteq \{\alpha + k_0\delta_0 + \delta_k, \alpha \in \Delta_{0,e_n}, k_0 \in \mathbb{Z}, k \in \mathbb{Z}^n\}$.

$$\tau_{\alpha+k_0\delta_0+\delta_k} = \mathfrak{g}(\overline{k}_0, \overline{k}, \alpha) \otimes t_0^{k_0} t^k \text{ for } \alpha \neq 0,$$
$$\tau_{k_0\delta_0+\delta_k} = \mathfrak{g}(\overline{k}_0, \overline{k}, 0) \otimes t_0^{k_0} t^k \oplus \bigoplus_{i=0}^{n} \mathbb{C} t_0^{k_0} t^k K_i \oplus \bigoplus_{i=0}^{n} \mathbb{C} t_0^{k_0} t^k d_i.$$

Notice that $\tau_0 = H$. Now we will define a non-degenerate bilinear form on H^*. For $\alpha \in \mathfrak{h}(0)^*$ extended α to H by $\alpha(K_i) = \alpha(d_i) = 0, 0 \leq i \leq n$.
Let $(\mathfrak{h}(0), K_i) = 0 = (\mathfrak{h}(0), d_i)$,
$(\delta_k + \delta_{k_0}, \delta_l + \delta_{l_0}) = 0 = (w_i, w_j)$,
$(\delta_i, w_j) = \delta_{ij}$. The form on $\mathfrak{h}(0)$ is the restriction of the form $(,)$ on $\mathfrak{g}$.

For $\gamma = \alpha + k_0\delta_0 + \delta_k$ is called real root if $\alpha \neq 0$ which is equivalent to $(\gamma, \gamma) \neq 0$. Denote Δ^{re} be the set of real roots. For $\alpha \in \Delta_{0,en}$, denote $\alpha^\vee$ the co-root of α.

Define $\gamma^\vee = \alpha^\vee + \frac{2}{(\alpha,\alpha)}\sum_{i=0}^{n} k_i K_i$ for γ real.

Then $\gamma(\gamma^\vee) = \alpha(\alpha^\vee) = 2$.

For γ real root, define reflection on H^* by

$$r_\gamma(\lambda) = \lambda - \lambda(\gamma^\vee)\gamma, \lambda \in H^*.$$

Let W be the Weyl group genarated by $r_\gamma, \gamma \in \Delta^{re}$.

Definition

A module V of τ is called integrable if

- $V = \bigoplus_{\lambda \in H^*} V_\lambda, V_\lambda = \{v \in V | hv = \lambda(h)v, h \in H\}, \dim V_\lambda < \infty,$
- $\mathfrak{g}(\overline{k}_0, \overline{k}, \alpha) \otimes t_0^{k_0} t^k$ acts locally nilpotently on V for $\alpha \neq 0$.

Let $P(V) = \{\gamma \in H^* | V_\gamma \neq 0\}$. For an irreducible integrable module with non zero central charge, we can assume that K_0 acts as $C_0 > 0$ and $K_i (i \neq 0)$ acts trivially upto a choice of co-ordinates. For any $\lambda \in P(V), \lambda(K_i) = C_i = 0$ for $1 \leq i \leq n$ and $\lambda(K_0) = C_0$. Let $\alpha_0 = -\beta_0 + \delta_0$ where β_0 is maximal root in $\Delta_{0,en}$. Note that α_0 may not be root of τ. Let $\alpha_1, \alpha_2, \cdots, \alpha_p$ be a set of simple roots for $\Delta(g(\overline{\circ}, \overline{\circ}), \mathfrak{h}(0))$ and let $Q^+ = \bigoplus_{i=0}^{p} \mathbb{N}\alpha_i$. Define an ordering on $H^*, \lambda \leq \mu$ for $\lambda, \mu \in H^*$, if $\mu - \lambda \in Q^+$.

Triangle decomposition

- We will now define triangular decomposition for τ. Let $Z = \Omega A/dA$. Let

$$
\begin{aligned}
L^+(\mathfrak{g},\sigma) &= \bigoplus_{\alpha+k_0\delta_0>0} \mathfrak{g}(\overline{k}_0,\overline{k},\alpha)\otimes t_0^{k_0}t^k, k\in\mathbb{Z}^n;\\
L^-(\mathfrak{g},\sigma) &= \bigoplus_{\alpha+k_0\delta_0<0} \mathfrak{g}(\overline{k}_0,\overline{k},\alpha)\otimes t_0^{k_0}t^k, k\in\mathbb{Z}^n;\\
L^0(\mathfrak{g},\sigma) &= \bigoplus_{k\in\mathbb{Z}^n} \mathfrak{g}(\overline{0},\overline{k},0)t^k;\\
D^+(m_0,m) &= \bigoplus_{\substack{0\leq i\leq n\\ s_0>0}} \mathbb{C}t_0^{s_0}t^s d_i, s\in\Gamma;\\
D^-(m_0,m) &= \bigoplus_{\substack{0\leq i\leq n\\ s_0<0}} \mathbb{C}t_0^{s_0}t^s d_i, s\in\Gamma;\\
D^0(m_0,m) &= \bigoplus_{0\leq i\leq n} \mathbb{C}t^s d_i, s\in\Gamma;
\end{aligned}
$$

$$
\begin{aligned}
Z^+ &= \bigoplus_{\substack{0\le i\le n\\ s_0>0}} \mathbb{C}t_0^{s_0}t^sK_i, s\in\Gamma;\\
Z^- &= \bigoplus_{\substack{0\le i\le n\\ s_0<0}} \mathbb{C}t_0^{s_0}t^sK_i, s\in\Gamma;\\
Z^0 &= \bigoplus_{0\le i\le n} \mathbb{C}t^sK_i, s\in\Gamma;\\
\tau^+ &= L^+(\mathfrak{g},\sigma)\oplus Z^+\oplus D^+(m_0,m);\\
\tau^- &= L^-(\mathfrak{g},\sigma)\oplus Z^-\oplus D^-(m_0,m);\\
\tau^0 &= L^0(\mathfrak{g},\sigma)\oplus Z^0\oplus D^0(m_0,m).
\end{aligned}
$$

Then clearly $\tau=\tau^-\oplus\tau^0\oplus\tau^+$ is a triangular decomposition.
Let $T=\{v\in V|\tau^+v=0\}\neq 0$ by Theorem 1.

(6.2) **Proposition:** T is a τ^0- module and in fact irreducible as τ^0-module. Further $V = U(\tau^-)T$.

Recall that $\{d_1, \cdots, d_n\} \subseteq D^0(m_0, m)$ and hence T is $\mathbb{Z}^n$- graded.
Let $T_k = \{v \in T | d_i v = (\lambda(d_i) + k_i)v, 1 \leq i \leq n\}$
where λ is a fixed weight in $P(V)$ coming from Theorem 1.

- It is easy to see that T can be identified with $V^1 \otimes A(m)$ where V^1 can be taken as
$$\bigoplus_{\substack{0 \le k_i < m_i \\ 1 \le i \le n}} \mathbf{T}_k$$
- Now $D^0(m_0, m)$ is spanned by $t^r d_i, r \in \Gamma, 0 \le i \le n$. Thus $D^0(m_0, m)$ can be identified with $DerA(m) \oplus \sum_{r \in \Gamma} \mathbb{C} t^r d_0$

  Z^0 can be identified with $\sum_{r \in \Gamma} \mathbb{C} t^r K_0$

  as the rest of the space acts trivially on T. Thus $V^1 \otimes A(m)$ is an irreducible module for
$$L = L^0(\mathfrak{g}, \sigma) \oplus DerA(m) \oplus \sum_{r \in \Gamma} \mathbb{C} t^r d_0 \oplus A(m),$$

- We note the following

$$\begin{aligned} t^r \cdot v \otimes t^s &= v \otimes t^{r+s}, \\ t^r d_0 \cdot v \otimes t^s &= \lambda(d_0) v \otimes t^{r+s} \text{ for } r, s \in \Gamma, v \in V^1. \end{aligned}$$

- Let $\overset{\circ}{\mathfrak{g}} = \{X \in \mathfrak{g} | \sigma_0 X = X, [h, X] = 0, h \in \mathfrak{h}(0)\}$
  the following is easily checked.
- $\sigma_i(\overset{\circ}{\mathfrak{g}}) \subseteq \overset{\circ}{\mathfrak{g}}$ for $1 \leq i \leq n$.
- $\overset{\circ}{\mathfrak{g}} = \bigoplus_{\overline{k} \in \Lambda} \overset{\circ}{\mathfrak{g}}_{\overline{k}}$ is a natural
  Λ- grading where $\overset{\circ}{\mathfrak{g}}_{\overline{k}} = \{X \in \overset{\circ}{\mathfrak{g}} | \sigma_i X = \xi_i^{k_i} X, 1 \leq i \leq n\}$

The corresponding multiloop algebra is denoted by

$$L(\overset{\circ}{\mathfrak{g}}, \sigma) = \bigoplus_{k \in \Lambda} \overset{\circ}{\mathfrak{g}}_{\overline{k}} \otimes t^k$$

It is clear that $L^0(\mathfrak{g}, \sigma) = L(\overset{\circ}{\mathfrak{g}}, \sigma)$.

When we say $X(k) = X \otimes t^k \in L(\overset{\circ}{\mathfrak{g}}, \sigma)$ we always mean $X \in \overset{\circ}{\mathfrak{g}}_{\overline{k}}$.

Thus $L \cong L(\overset{\circ}{\mathfrak{g}}, \sigma) \oplus DerA(m) \oplus A(m) \oplus \sum_{r \in \Gamma} \mathbb{C} t^r d_0$.

- The brackets in L are given as follows :
- $[X(k), Y(l)] = [X, Y](k + l)$,
- $[D(u, r), D(v, s)] = D(w, r + s)$ where $w = (u, s)v - (v, r)u$,
- $[D(u, r), t^s] = (u, s)t^{r+s}$,
- $[D(u, r), X(k)] = (u, k)X(k + r)$,
- $[D(u, r), t^s d_0] = (u, s)t^{r+s} d_0$.

Now we would like to classify the irreducible L- module $V^1 \otimes A(m)$.

We recall some facts from Rao on $DerA(m)$. Let $I(u,r) = D(u,r) - D(u,0), u \in \mathbb{C}^n, r \in \Gamma$. It is easy to check, $[I(u,r), I(v,s)] = (v,r)I(u,r) - (u,s)I(v,s) + I(w,s+r)$ where $w = (u,s)v - (v,r)u$ Let I be the space spanned by $I(u,r), u \in \mathbb{C}^n, r \in \Gamma$ which can be seen as subalgebra of $DerA(m)$.

- Let W be the subspace of $V^1 \otimes A(m)$ spanned by vectors of the form $t^r.v(s) - v(s)$ for $r, s \in \Gamma$ and $v \in V^1$. Let $\widetilde{L} = I \ltimes L(\overset{0}{\mathfrak{g}}, \sigma)$

Lemma

W is an $\widetilde{L} \oplus A(m) \oplus \sum_{r\in\Gamma} \mathbb{C}t^r d_0$ module.

Let $\widetilde{V} = (V^1 \otimes A(m))/W$ which is an $\widetilde{L}$- module. Notice that $A(m) \oplus \sum_{r\in\Gamma} \mathbb{C}t^r d_0$ acts as scalars on $\widetilde{V}$ and hence we ignore them.

We would like to prove that $\widetilde{V}$ is completely reducible $\widetilde{L}$-module.

1. Recall that the Lie brackets in $\widetilde{L}$ are given by
2. $[I(v,s), I(u,r)] = (u,s)I(v,s) - (v,r)I(u,r) + I(w, r+s)$, where $I(u,r) = D(u,r) - D(u,0)$ and $w = (v,r)u - (u,s)v$
3. $[I(v,s), X(k)] = (v,k)(X(s+k) - X(k))$,
4. $[X(k), Y(l)] = [X,Y](k+l)$,

where $X \in \mathfrak{g}_{\overline{k}}$, $Y \in \mathfrak{g}_{\overline{l}}$, $k, l \in \mathbb{Z}^n$, $r, s \in \Gamma$ and $u, v \in \mathbb{C}^n$.

- Recall that we fixed $\lambda \in P(V)$. Let $\alpha_i = \lambda(d_i)$ Let $\alpha = \sum \alpha_i e_i \in \mathbb{C}^n$ and let V_1 is an $\tilde{L}$- module. Then we will define L- module structure on $L(V_1) = V_1 \otimes A_n$.

$$
\begin{aligned}
X(k) \cdot v_1 \otimes t^l &= (X(k)v_1) \otimes t^{l+k}, \\
D(u,r) \cdot v_1 \otimes t^l &= (I(u,r)v_1) \otimes t^{l+r} + (u, l+\alpha) v_1 \otimes t^{l+r}, \\
t^s v_1 \otimes t^l &= v_1 \otimes t^{s+l}, \\
t^r d_0 \cdot v_1 \otimes t^l &= \lambda(d_0) \cdot v_1 \otimes t^{l+r},
\end{aligned}
$$

where $v_1 \in V_1, l, k \in \mathbb{Z}^n, r, s, \in \Gamma$.

So

- T is an irreducible L-module.
- $\widetilde{V}$ is an $\widetilde{L}$-module.
- $L(\widetilde{V})$ is an L-module.

We will now establish that T is contained in $L(\widetilde{V})$ as L-modules.
For $v_k \in T_k$, let $\overline{v}_k$ be the image in $\widetilde{V} \cong T/W$.
Let $\widetilde{\varphi}$: $T \to L(\widetilde{V})$, $\widetilde{\varphi}\,(v_k) = \overline{v}_k \otimes t^k, k \in \mathbb{Z}^n$.

Lemma

$\widetilde{\varphi}$ *is an L-module map.*

Theorem

$\widetilde{V}$ is completely reducible as $\widetilde{L}$- module and all components are isomorphic.

$$\tilde{V} = \bigoplus_{\bar{p}\in\Lambda} \tilde{V}_{\bar{p}}$$

Let $p \in \mathbb{Z}^n$ and $\bar{p} \in \Lambda$. Define

$$L(\tilde{V})(\bar{p}) = \{\bar{v}_k \otimes t^{k+r+p}, \bar{v}_k \in \tilde{V}_{\bar{k}}, r \in \Gamma, k \in \mathbb{Z}^n\}$$

clearly

$$L(\tilde{V}) = \bigoplus_{\bar{p}\in\Lambda} L(\tilde{V})(\bar{p})$$

which is a finite sum of L-modules.

Lemma

$\widetilde{V}$ is graded irreducible $\widetilde{L}$-module if and only if $L(\widetilde{V})(0)$ is an irreducible L-module.

It is to see that
$T \cong L(\widetilde{V})(\overline{0})$ as L-modules and in particular $L(\widetilde{V})(\overline{0})$ is an irreducible L-module. Thus by above Lemma $\widetilde{V}$ is irreducible graded $\widetilde{L}$-module.

Proposition

Each $L(\widetilde{V})(\overline{p})$ is an irreducible L-module.

Theorem

$\widetilde{V}$ is an $\widetilde{L}$ irreducible module if and only if $L(\widetilde{V})(\overline{p}), \overline{p} \in \Lambda$ are mutually non-isomorphic as L-modules.

We have seen that $T \cong L(\widetilde{V})(0)$ as L-modules. Let $M = Ind^{\tau}_{\tau_0+\tau^+} T$ Then there exists a unique maximal submodule M^{rad} intersecting T trivially. Thus M/M^{rad} is irreducible and isomorphic to the original module V.

Theorem

Let V be an irreducible integrable module for τ with K_0 acts as $C_0 > 0$ and K_i acts trivially. Let T and M as above. Then $V \cong M/M^{rad}$ as τ-modules.

Happy 60^{th} Birthday to Chari

Thank you